

The Algebraic Curve of Classical Superstrings on $AdS_5 \times S^5$

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Abstract

We investigate the monodromy of the Lax connection for classical IIB superstrings on $AdS_5 \times S^5$. For any solution of the equations of motion we derive a spectral curve of degree $4 + 4$. The curve consists purely of conserved quantities, all gauge degrees of freedom have been eliminated in this form. The most relevant quantities of the solution, such as its energy, can be expressed through certain holomorphic integrals on the curve. This allows for a classification of finite gap solutions analogous to the general solution of strings in flat space. The role of fermions in the context of the algebraic curve is clarified. Finally, we derive a set of integral equations which reformulates the algebraic curve as a Riemann-Hilbert problem. They agree with the planar, one-loop $\mathcal{N} = 4$ supersymmetric gauge theory proving the complete agreement of spectra in this approximation.

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1 Introduction and Overview

Strings in flat space have been solved a long time ago. The solution of the classical equations of motion is straight-forward and obtained by a Fourier transformation, or mode decomposition, of the world sheet. The string is then represented by a collection of independent harmonic oscillators, one for each mode and orientation in target space. The oscillators are merely coupled by the Virasoro and level-matching constraints. The conserved, physical quantities of the string are the absolute values of oscillator amplitudes. Quantization of this system essentially poses no problem. The harmonic oscillators are excited in quanta and the amplitudes turn into integer-valued excitation numbers.

Maldacena's conjecture [1] however brought about special attention on strings in curved target spaces with 'RR-flux', in particular IIB superstrings on $AdS_5 \times S^5$. There, a solution and quantization is much more involved due to the highly non-linear nature of the string action [2]. A direct quantization of the world sheet theory is furthermore obstructed by conformal and kappa symmetry which require gauge fixing. This introduces a number of additional terms and usually makes the problem intractable.

One path to quantization is related to the maximally supersymmetric plane-wave background [3] and the correspondence to gauge theory [4]. In this background the solution and quantization closely resembles its flat space counterpart [5]. The full $AdS_5 \times S^5$ background may be regarded as a deformation of plane waves. Following this idea, one can obtain a quantum string on $AdS_5 \times S^5$ in a perturbation series around plane waves [6]. This approach has yielded several important insights into the quantum nature of the string, but there are drawbacks: The perturbative expansion is very involved, the first order is feasible [7], but beyond there are no definite answers available yet. Even if this problem might be overcome, still we would be limited to a certain region of the parameter space of full $AdS_5 \times S^5$ which is insensitive to global aspects.

Another approach to strings in curved space is to consider classical solutions, see e.g. [8]. For these solutions with large spins one can show that quantum effects are suppressed and already the classical solution yields a good approximation for the full energy. Even more excitingly, Frolov and Tseytlin discovered that many of these spinning string solutions have an expansion which is in qualitative agreement with the loop expansion of gauge theory [9]. Their conjecture of a quantitative agreement has been confirmed in several cases in [10, 11] and many more works since,¹ see [13–16] for reviews of the subject. Finding exact solutions is not trivial, the complexity of the functions increases with the complexity of the solution. The functions that occur are of algebraic, elliptic or hyperelliptic type and many of those which can be expressed using conventional functions have been found. While in principle each and every solution can be found using suitable (unconventional) functions, it is impossible to catalog infinitely many of them in order to understand their generic structure.

Finding the energy spectrum of superstrings on $AdS_5 \times S^5$ therefore appears a too difficult problem to be solved explicitly. Instead one can ask a more moderate question: How is the spectrum of string solutions organized? In other words, can we classify string

¹Here, as well as in the case of near plane-wave strings there are discrepancies starting at three gauge theory loops [6, 12]. This puzzle can also be reformulated as the question why it works at one and two loops in the first place. We have little to add on this issue.

solutions even though we cannot write them explicitly? Understanding the classification at the classical level might be an essential step towards understanding the quantum string. The classification was started in [17] for bosonic strings on $\mathbb{R} \times S^3$ which is a subspace of the full $AdS_5 \times S^5$ background. It was shown that for each solution of the equations of motion there exists a corresponding hyperelliptic curve. The key physical data of the solution, such as the energy and Noether charges, were identified in the algebraic curve.²

At this point one can turn the logic around and investigate the moduli space of admissible curves, i.e. those curves which correspond to some classical solution. This leads to a solution of the spectral problem in terms of algebraic curves, which is probably as close to an explicit solution as it can be. However, one would have to ensure that all relevant constraints on the structure of admissible curves have been correctly identified. A survey of the moduli space of admissible curves suggests that this is indeed the case: There turns out to be one continuous modulus per genus and each handle of the curve can be interpreted as a particular string mode. This count matches with strings in flat space, which has one amplitude per string oscillator. Although two distinct theories are compared here, one can expect that the number of local degrees of freedom of the string should be independent of the background. We furthermore believe that the (conserved) moduli of a curve represent a complete set of action variables for the string. The moduli space of admissible curves would thus represent half the phase space of the string model.

Another interesting option is to reformulate the problem of finding admissible curves as a Riemann-Hilbert problem. This is achieved by representing the curve as a collection of Riemann sheets connected by branch cuts. The branch cuts are represented by integrals over contours and densities in the complex plane. The admissibility conditions turn into integral equations on these contours and densities. This representation reveals an underlying scattering problem and the branch cuts represent the fundamental particles. The integral equations select equilibrium states of the scattering problem. This can be compared to a direct Fourier transformation of the string: The Fourier transformation transforms the equations of motion into equations among the different Fourier modes. Conceptually, the resulting equations are very similar to the integral equations. The main difference between the two approaches is that there are interactions between arbitrarily many Fourier modes due to the highly non-linear nature of the strings, while the interactions for the integral equations are only *pairwise*! In some sense, the algebraic curve can thus be interpreted as a clever mode decomposition specifically tailored for the particular curved background.

The pairwise, i.e. factorized, nature of the scattering problem leads us to integrability. Indeed, the algebraic curve was constructed using the Lax connection, a family of flat connections on the two-dimensional string world sheet. For sigma models on group manifolds and symmetric coset spaces, such as $SU(2) = S^3$, this connection is well-known [19] and related to integrability as well as an infinite set of conserved charges [20] of the two-dimensional theory. Integrable structures were also found in the AdS/CFT

²This explains, among other things, why the classical energy, one of these charges, is typically expressed through hyperelliptic functions. The various integration constants of the classical solution turn into moduli of the algebraic curve which appear as parameters to the hyperelliptic functions. See also [18] for a discussion of the moduli of some particular curves.

dual $\mathcal{N} = 4$ gauge theory: The dual of the world-sheet Hamiltonian, the planar dilatation operator (see [15] for a review), was shown to be integrable at leading loop order [21, 22]. Moreover, there are indications that integrability is not broken by higher-loop effects [23]. In gauge theory, integrability enables one to construct a Bethe ansatz [24] to diagonalize local operators, which are isomorphic to quantum spin chains. This leads to a set of algebraic equations [21, 22, 12, 25] whose solutions are in one-to-one correspondence to eigenstates of the dilatation operator. In the limit of states with a large number of partons, which is at the heart of the spinning-strings correspondence, the discrete Bethe equations turn into integral equations [26, 10]. These are very similar to the integral equations from the string sigma model. In fact, it was shown that the higher-loop Bethe equations in the $\mathfrak{su}(2)$ sector [12, 25] match with the equations from classical string theory on $\mathbb{R} \times S^3$ up to two gauge-theory loops [17]. This proves the equality of energy spectra in this limit and sector. Alternatively, one can also derive an algebraic curve for gauge theory and compare it to the one for the sigma model [17]. An altogether different approach to showing the agreement of spectra uses coherent states [27].

The solution of the spectral problem in terms of algebraic curves has since been extended to three other subsectors of the full superstring: Bosonic strings on $AdS_3 \times S^1$ [28], on $\mathbb{R} \times S^5$ [29] and on $AdS_5 \times S^1$ [30]. Also some features of the assembly of full AdS_5 and full S^5 are known [31]. In all previous analyses, however, *fermions* have been excluded.³ While this is justified (for almost all practical purposes) at the classical level, they are certainly required to give a consistent quantum theory. It is therefore essential to include them, even at the classical level. This can indeed be done, even though it is a classical setting.⁴

In the present article we shall derive the solution of the spectrum of IIB superstrings on $AdS_5 \times S^5$ in terms of algebraic curves. The starting point will be the family of flat connections found by Bena, Polchinski and Roiban [34]. Using its open Wilson loop around the closed-string world sheet, the so-called monodromy, we can derive an algebraic curve. As was demonstrated in [34] the Lax connection exists prior to gauge fixing. We therefore do not fix any gauge, neither of conformal nor of kappa symmetry, in contrast to [17, 28, 29] and especially [31]. The emergent curve is neither a regular algebraic curve, nor an algebraic supercurve, i.e. not a supermanifold. It almost splits in two parts, but it is held together by the fermions. Each part has degree four and corresponds to one of the S^5 and AdS_5 coset models. The bosonic degrees of freedom give rise to square-root branch points and cuts connecting them. These appear only within each set of four Riemann sheets. We shall show that, conversely, the fermions give rise to poles. Poles come in pairs, one of them is on the S^5 -part of the curve, the other on the AdS_5 -part while their residues are the same.⁵ Their position within the

³Within the Frolov-Tseytlin correspondence fermions have been treated in [32, 33] using the coherent state approach.

⁴Having fermions in classical *equations* is not a problem, but we run into difficulties when we try to find explicit *solutions*, which would require the introduction of actual Grassmann numbers.

⁵The residues are products of two Grassmann-odd numbers. Therefore, they are Grassmann-even, but not of zeroth degree, i.e. they cannot be represented by common numbers. This is why fermions can be neglected for almost all practical purposes. The derivations are however simplified by ignoring this fact.

algebraic curve is determined by the bosonic background. The two parts of the curve are furthermore linked by the Virasoro constraint: It relates a set of fixed poles between the two parts of the curve. These poles are an important general characteristics of the model and do not correspond to fermions.

The precise structure of the algebraic curve and its representation in the form of integral equations constitute the key information from string theory for a comparison with gauge theory [35, 22, 15, 30] via the Frolov-Tseytlin proposal. Using an integral representation for the curve, we are able to show agreement of the spectra at leading order in the effective coupling constant. A more detailed comparison will be performed in the follow-up article [36].

The structure of this article is as follows: In Sec. 2 we will investigate the monodromy of the Lax connection and derive an algebraic curve from it. The remainder of the section is devoted to finding the analytic properties of the curve and relating them to data of the associated string solution. Then we decouple from the underlying string solution in Sec. 3 and consider the set of admissible curves. After counting the number of moduli, we shall identify them with certain integrals on the curve. Their relationship to the global charges is established. In the final Sec. 4 we shall represent the algebraic curve by means of its branch cuts between the Riemann sheets. The resulting equations are closely related to the equations one obtains from spin chains in the thermodynamic limit. We show that they agree with one-loop gauge theory. We conclude and give an outlook in Sec. 5. The appendices contain a review of supermatrices (App. A), the relation between coset and vector models (App. B) and explicit but lengthy expressions related to the full supersymmetric sigma model (App. D).

2 Supersymmetric Sigma Model

We start by investigating the $AdS_5 \times S^5$ supersymmetric sigma model on a closed string worldsheet. First of all we present the sigma model in terms of its fields, currents and constraints. Then we review the Lax connection and its monodromy and show that the essential physical information (action variables) is described by an algebraic curve. The remainder of this section is devoted to special properties of the curve and relating them to physical quantities.

The $AdS_5 \times S^5$ superspace can be represented as the coset space of the supergroup $PSU(2, 2|4)$ over $Sp(1, 1) \times Sp(2)$. Up to global issues, but preserving the algebraic structure, we can change the signature of the target spacetime. Here we will consider the coset $PSL(4|4, \mathbb{R})/Sp(4, \mathbb{R}) \times Sp(4, \mathbb{R})$. This choice is convenient as we can completely avoid complex conjugation which may be somewhat confusing, especially in a supersymmetric setting. See e.g. [37, 38] for an explicit treatment of the $PSU(2, 2|4)$ coset model. The global issues that we should keep in mind are whether the string can wind around the manifold. For S^5 this is certainly the case, while for AdS_5 there should be no windings. Note that the physical AdS_5 is a universal cover and there cannot be windings around the unfolded time circle. Likewise, the involved group manifolds are considered to be universal coverings.

2.1 The Coset Model

The Metsaev-Tseytlin string is a coset space sigma model. To represent the coset, we consider a group element g of $\text{PSL}(4|4, \mathbb{R})$ and two constant $(4|4) \times (4|4)$ matrices⁶

$$E_1 = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array} \right), \quad E_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E \end{array} \right), \quad (2.1)$$

which break $\text{PSL}(4|4, \mathbb{R})$ to $\text{Sp}(4, \mathbb{R}) \times \text{Sp}(4, \mathbb{R})$. Here, E is an antisymmetric 4×4 matrix⁷

$$E = \left(\begin{array}{cc} 0 & +I \\ -I & 0 \end{array} \right), \quad (2.2)$$

where each entry corresponds to a 2×2 block and I is the identity matrix. We shall denote the pseudo-inverses of E_1, E_2 by

$$\bar{E}_1 = \left(\begin{array}{c|c} E^{-1} & 0 \\ \hline 0 & 0 \end{array} \right), \quad \bar{E}_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E^{-1} \end{array} \right). \quad (2.3)$$

These are defined such that a product of E_a and \bar{E}_b is a projector to the even/odd subspace if $a = b$ or zero if $a \neq b$. Finally, let us introduce a grading matrix

$$\eta = \left(\begin{array}{c|c} +I & 0 \\ \hline 0 & -I \end{array} \right) \quad (2.4)$$

which will be useful at various places.

The breaking of $\text{PSL}(4|4, \mathbb{R})$ to $\text{Sp}(4, \mathbb{R}) \times \text{Sp}(4, \mathbb{R})$ is achieved as follows: The matrix E_1 is invariant under $E_1 \mapsto h E_1 h^{\text{ST}}$ for elements h of a subgroup $\text{Sp}(4, \mathbb{R}) \times \text{SL}(4, \mathbb{R})$ of $\text{PSL}(4|4, \mathbb{R})$. Similarly, $E_2 \mapsto h E_2 h^{\text{ST}}$ is invariant under a $\text{SL}(4, \mathbb{R}) \times \text{Sp}(4, \mathbb{R})$. The combined map $(E_1, E_2) \mapsto (h E_1 h^{\text{ST}}, h E_2 h^{\text{ST}})$ leaves (E_1, E_2) invariant precisely for $h \in \text{Sp}(4, \mathbb{R}) \times \text{Sp}(4, \mathbb{R})$. Thus the element $(g E_1 g^{\text{ST}}, g E_2 g^{\text{ST}})$ with $g \in \text{PSL}(4|4, \mathbb{R})$ parametrizes the $AdS_5 \times S^5$ superspace.⁸

We now introduce the supermatrix-valued field $g(\tau, \sigma) \in \text{PSL}(4|4, \mathbb{R})$ on the world-sheet. It satisfies $\text{sdet } g = 1$ and we identify group elements which are related by an abelian rescaling, $g \hat{=} \xi g$.⁹ The field g is not necessarily strictly periodic but

$$g(\tau, \sigma + 2\pi) = g(\tau, \sigma) h(\tau, \sigma). \quad (2.5)$$

with $h(\tau, \sigma)$ an element of $\text{Sp}(4, \mathbb{R}) \times \text{Sp}(4, \mathbb{R})$. We define the standard g -connection J as

$$J = -g^{-1} dg. \quad (2.6)$$

⁶For a short review of the algebra of supermatrices, cf. App. A.

⁷In fact, any $E = -E^{\text{T}}$ with $\varepsilon_{\alpha\beta\gamma\delta} E^{\alpha\beta} E^{\gamma\delta} \neq 0$ would suffice and one could as well pick distinct matrices E for E_1 and E_2 .

⁸Note that E_1 is an antisymmetric supermatrix, $E_1^{\text{ST}} = -\eta E_1$, while E_2 is symmetric, $E_2^{\text{ST}} = +\eta E_2$. Therefore also $g(E_1 \pm E_2)g^{\text{ST}}$ or $g(E_1 \pm i E_2)g^{\text{ST}}$ parametrize the coset as we can disentangle the contributions from E_1 and E_2 by projecting to the symmetric and antisymmetric parts.

⁹For $(4|4) \times (4|4)$ supermatrices, $\text{sdet } \xi I = 1$ for any number ξ .

It is flat and supertraceless

$$dJ = J \wedge J \quad \text{and} \quad \text{str } J = 0 \quad (2.7)$$

by means of the usual identities and $\text{sdet } g = 1$. The algebra $\mathfrak{psl}(4|4, \mathbb{R})$ can be decomposed into four parts obeying a \mathbb{Z}_4 -grading [2, 39, 40].¹⁰ The connection decomposes as follows

$$J = H + Q_1 + P + Q_2. \quad (2.8)$$

We use the constant supermatrices $E_{1,2}, \bar{E}_{1,2}$ to project to the various components

$$\begin{aligned} H &= \frac{1}{2} E_1 \bar{E}_1 J E_1 \bar{E}_1 - \frac{1}{2} E_1 J^{\text{ST}} \bar{E}_1 + \frac{1}{2} E_2 \bar{E}_2 J E_2 \bar{E}_2 - \frac{1}{2} E_2 J^{\text{ST}} \bar{E}_2, \\ Q_1 &= \frac{1}{2} E_1 \bar{E}_1 J E_2 \bar{E}_2 + \frac{1}{2} E_1 J^{\text{ST}} \bar{E}_2 + \frac{1}{2} E_2 \bar{E}_2 J E_1 \bar{E}_1 - \frac{1}{2} E_2 J^{\text{ST}} \bar{E}_1, \\ P &= \frac{1}{2} E_1 \bar{E}_1 J E_1 \bar{E}_1 + \frac{1}{2} E_1 J^{\text{ST}} \bar{E}_1 + \frac{1}{2} E_2 \bar{E}_2 J E_2 \bar{E}_2 + \frac{1}{2} E_2 J^{\text{ST}} \bar{E}_2, \\ Q_2 &= \frac{1}{2} E_1 \bar{E}_1 J E_2 \bar{E}_2 - \frac{1}{2} E_1 J^{\text{ST}} \bar{E}_2 + \frac{1}{2} E_2 \bar{E}_2 J E_1 \bar{E}_1 + \frac{1}{2} E_2 J^{\text{ST}} \bar{E}_1. \end{aligned} \quad (2.9)$$

They satisfy the \mathbb{Z}_4 -graded Bianchi identities in [34]

$$\begin{aligned} dH &= H \wedge H + Q_1 \wedge Q_2 + P \wedge P + Q_2 \wedge Q_1, \\ dQ_1 &= H \wedge Q_1 + Q_1 \wedge H + P \wedge Q_2 + Q_2 \wedge P, \\ dP &= H \wedge P + Q_1 \wedge Q_1 + P \wedge H + Q_2 \wedge Q_2, \\ dQ_2 &= H \wedge Q_2 + Q_1 \wedge P + P \wedge Q_1 + Q_2 \wedge H, \end{aligned} \quad (2.10)$$

and their supertraces vanish

$$\text{str } H = \text{str } Q_1 = \text{str } P = \text{str } Q_2 = 0. \quad (2.11)$$

Note that $\text{str } H = \text{str } Q_1 = \text{str } Q_2 = 0$ is satisfied by means of the projections (2.9) while $\text{str } P = \text{str } J = 0$ holds due to (2.7).

The action of the IIB superstring on $AdS_5 \times S^5$ given in [2] in terms of the connections $P, Q_{1,2}$ reads [40]

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int \left(\frac{1}{2} \text{str } P \wedge *P - \frac{1}{2} \text{str } Q_1 \wedge Q_2 + \Lambda \wedge \text{str } P \right). \quad (2.12)$$

We have introduced the Lagrange multiplier Λ to enforce $\text{str } P = 0$. In fact, we cannot remove the part proportional to the identity matrix because of the identity $\text{str } I = 0$. The equations of motion read

$$\begin{aligned} 0 &= P \wedge Q_2 - *P \wedge Q_2 + Q_2 \wedge P - Q_2 \wedge *P, \\ d*P &= H \wedge *P + Q_1 \wedge Q_1 + *P \wedge H - Q_2 \wedge Q_2 + d\Lambda, \\ 0 &= P \wedge Q_1 + *P \wedge Q_1 + Q_1 \wedge P + Q_1 \wedge *P. \end{aligned} \quad (2.13)$$

The appearance of Λ in the equations of motion is related to the projective identification $g \hat{=} \xi g$. The equations of motion can also be written as the g -covariant conservation of the global $\mathfrak{psl}(4|4, \mathbb{R})$ symmetry current K

$$d*K - J \wedge *K - *K \wedge J = 0, \quad K = P + \frac{1}{2} *Q_1 - \frac{1}{2} *Q_2 - *\Lambda. \quad (2.14)$$

¹⁰The \mathbb{Z}_4 -grading is directly related to supertransposing, c.f. App. A.

The above equations of motion follow after decomposition into the \mathbb{Z}_4 -graded components. The dependence of K on the unphysical Lagrange multiplier reflects the ambiguity in the definition of the abelian part of K in $\mathfrak{psl}(4|4, \mathbb{R})$. In the fixed frame,¹¹ which is related to the moving one by $k = gKg^{-1}$, the equations for the current are even shorter

$$d*k = 0. \quad (2.15)$$

The global symmetry charges are consequently given by

$$s = \frac{\sqrt{\lambda}}{2\pi} \oint_{\gamma} *k = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma k_{\tau}. \quad (2.16)$$

These do not depend on the form of the path γ around the closed loop and are thus conserved physical quantities. For later convenience, we rewrite $s = g(0)Sg^{-1}(0)$ in terms of the moving-frame current K as follows

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma g^{-1}(0)g(\sigma)K_{\tau}(\sigma)g^{-1}(\sigma)g(0) \\ &= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \left(\text{P exp} \int_0^{\sigma} d\sigma' J_{\sigma}(\sigma') \right)^{-1} K_{\tau}(\sigma) \left(\text{P exp} \int_0^{\sigma} d\sigma' J_{\sigma}(\sigma') \right). \end{aligned} \quad (2.17)$$

Here, as for the remainder of the article, the path ordering symbol P puts the values of σ in decreasing order from left to right.

In addition to the equation of motion, the Virasoro constraints following from variation of the world-sheet metric (which appears only within the dualization $*$) are given by

$$\text{str } P_{\pm}^2 = 0. \quad (2.18)$$

Here we have introduced the light-cone coordinates

$$\sigma_{\pm} = \frac{1}{2}(\tau \pm \sigma), \quad \partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}, \quad P_{\pm} = P_{\tau} \pm P_{\sigma}. \quad (2.19)$$

2.2 Lax Connection and Monodromy

A family of flat connections $a(\kappa)$ for the superstring on $AdS_5 \times S^5$ was derived in [34].¹² This was expressed in the fixed frame, which is related to moving one by $j = gJg^{-1}$ and similarly for H, Q_1, P, Q_2 . The Lax connection is given by

$$a(\kappa) = \alpha(\kappa) p + \beta(\kappa) (*p - A) + \gamma(\kappa) (q_1 + q_2) + \delta(\kappa) (q_1 - q_2) \quad (2.20)$$

¹¹We shall distinguish between a moving frame and a fixed frame. In the moving frame E is a constant matrix and the fundamental field is g . The gauge connection is $D = d - J$. In the fixed frame the matrix corresponding to E is gEg^{5T} . It is not constant but rather the fundamental field. The gauge connection is trivial, $D = d$. See App. B for a comparison of both formalisms. We use uppercase and lowercase letters for the moving and fixed frames, respectively.

¹²For complex values of κ , there is only one family of flat connections. The other family mentioned in [34] is trivially obtained by replacing κ with $i\pi - \kappa$.

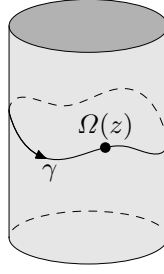


Figure 1: The monodromy $\Omega(z)$ is the open Wilson loop of the Lax connection $A(z)$ around the string.

with

$$\begin{aligned}\alpha(\kappa) &= -2 \sinh^2 \kappa, & \gamma(\kappa) &= 1 - \cosh \kappa, \\ \beta(\kappa) &= 2 \sinh \kappa \cosh \kappa, & \delta(\kappa) &= \sinh \kappa.\end{aligned}\tag{2.21}$$

We will employ a more convenient parametrization by setting $z = \exp \kappa$. The coefficient functions become

$$\begin{aligned}\alpha(z) &= 1 - \frac{1}{2}z^2 - \frac{1}{2}z^{-2}, & \gamma(z) &= 1 - \frac{1}{2}z - \frac{1}{2}z^{-1}, \\ \beta(z) &= \frac{1}{2}z^2 - \frac{1}{2}z^{-2}, & \delta(z) &= \frac{1}{2}z - \frac{1}{2}z^{-1}.\end{aligned}\tag{2.22}$$

We would now like to transform the connection to the moving frame using $J = g^{-1}jg$ and compute

$$\begin{aligned}d - A(z) &= g^{-1}(d + a(z))g = d - J + g^{-1}a(z)g \\ &= d - H + (\alpha - 1)P + \beta(*P - \Lambda) + (\gamma - 1)(Q_1 + Q_2) + \delta(Q_1 - Q_2),\end{aligned}\tag{2.23}$$

where the Lax connection reads

$$A(z) = H + \left(\frac{1}{2}z^2 + \frac{1}{2}z^{-2}\right)P + \left(-\frac{1}{2}z^2 + \frac{1}{2}z^{-2}\right)(*P - \Lambda) + z^{-1}Q_1 + zQ_2.\tag{2.24}$$

As was shown in [34], it satisfies the flatness condition

$$(d - A(z))^2 = 0\tag{2.25}$$

by means of the equations of motion. It is also traceless for obvious reasons, $\text{str } A(z) = 0$.

As emphasized in [17, 29], an important object for the solution of the spectral problem is the open Wilson loop of the Lax connection around the closed string. It is given by

$$\Omega_0(z) = \text{P exp} \int_0^{2\pi} d\sigma A_\sigma(z) \simeq \text{P exp} \oint A(z).\tag{2.26}$$

The *monodromy* which is defined as¹³

$$\Omega(z) = \Omega_0^{-1}(1) \Omega_0(z)\tag{2.27}$$

¹³For $z = 1$ the Lax connection $A(z) = J$ is the gauge connection. The additional factor $\Omega_0^{-1}(1) = g(0)^{-1}g(2\pi) = h(0)$ therefore transforms the monodromy back to the tangent space at $\sigma = 0$.

is independent of the path γ around the closed string; it merely depends on the point $\gamma(2\pi) = \gamma(0)$ where the path is cut open. More explicitly, a shift of $\gamma(0)$ leads to a similarity transformation (\simeq), see e.g. [29]. Therefore, the eigenvalues of $\Omega(z)$ are invariant, physical quantities. Note that we did not specify any particular gauge of conformal or kappa symmetry. Under kappa symmetry the Lax connection transforms by conjugation [41] and consequently leaves the eigenvalues invariant as well. For definiteness we define $\Omega(z)$ through the path $\sigma \in [0, 2\pi]$ at $\tau = 0$. Also note that $\text{str } A(z) = 0$ leads to $\text{sdet } \Omega(z) = 1$.

In the Hamiltonian formulation, the eigenvalues of the monodromy represent action variables of the sigma model.¹⁴ We have a one-parameter family of them and it is not inconceivable that they form a complete set. So we might have a sufficient amount of information to fully characterize the class of solution. The time-dependent angle variables and all gauge degrees of freedom are completely projected out in the eigenvalues of $\Omega(z)$. This is a very good starting point for a quantum theory: For quantum eigenstates we can measure all the action variables exactly but information of the angle variables is obscured by the uncertainty principle.

2.3 The Algebraic Curve

The physical information of the monodromy matrix is its conjugation class. Let $u(z)$ diagonalize $\Omega(z)$ as follows

$$u(z)\Omega(z)u^{-1}(z) = \text{diag} \{e^{i\tilde{p}_1(z)}, e^{i\tilde{p}_2(z)}, e^{i\tilde{p}_3(z)}, e^{i\tilde{p}_4(z)} || e^{i\hat{p}_1(z)}, e^{i\hat{p}_2(z)}, e^{i\hat{p}_3(z)}, e^{i\hat{p}_4(z)}\}. \quad (2.28)$$

Note that the eigenvalues $e^{i\tilde{p}_k}$ and $e^{i\hat{p}_l}$ corresponding to the two gradings are distinguishable, they cannot be interchanged by a (bosonic) similarity transformation. We can associate \tilde{p}_k to S^5 while \hat{p}_k corresponds to AdS^5 . In contrast, we may freely interchange eigenvalues within each set of four. Unimodularity, $\text{sdet } \Omega(z) = 1$, translates to the condition

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 - \hat{p}_1 - \hat{p}_2 - \hat{p}_3 - \hat{p}_4 \in 2\pi\mathbb{Z}. \quad (2.29)$$

The monodromy (2.27) depends analytically on the spectral parameter z by definition except at the singular points $z = 0$ and $z = \infty$. This however does not imply that also the eigenvalues $\{e^{i\tilde{p}_k} || e^{i\hat{p}_k}\}$ enjoy the same property.

Let us first consider a point \tilde{z}_a where two eigenvalues $e^{i\tilde{p}_k}, e^{i\hat{p}_l}$ corresponding to the S^5 -part of the sigma model degenerate. The restriction of $\Omega(z)$ to the subspace of the two corresponding eigenvalues then takes the general form

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.30)$$

with some coefficients a, b, c, d depending analytically on z . Its eigenvalues are given by the general formula

$$\gamma_{1,2} = \frac{1}{2} \left(a + d \pm \sqrt{(a-d)^2 + 4bc} \right). \quad (2.31)$$

¹⁴See [38] for an investigation of the Poisson brackets of the monodromy.

At $z = \tilde{z}_a$ the combination $f = (\gamma_1 - \gamma_2)^2 = (a - d)^2 + 4bc = (\text{Tr } \Gamma)^2 - 2 \text{Tr } \Gamma^2$ under the square root vanishes, $f(\tilde{z}_a) = 0$. In the generic case, one can expect $f'(\tilde{z}_a) \neq 0$. This implies the well-known fact that crossing of eigenvalues usually gives rise to a square-root singularity:

$$e^{i\tilde{p}_{k,l}(z)} = e^{i\tilde{p}_k(\tilde{z}_a)} \left(1 \pm \tilde{\alpha}_a \sqrt{z - \tilde{z}_a} + \mathcal{O}(z - \tilde{z}_a) \right). \quad (2.32)$$

Similarly, coincident AdS_5 -eigenvalues $e^{i\tilde{p}_k}$ and $e^{i\tilde{p}_l}$ at \hat{z}_a lead to square-root singularities

$$e^{i\hat{p}_{k,l}(z)} = e^{i\hat{p}_k(\hat{z}_a)} \left(1 \pm \hat{\alpha}_a \sqrt{z - \hat{z}_a} + \mathcal{O}(z - \hat{z}_a) \right). \quad (2.33)$$

The behavior around a point z_a^* where eigenvalues of opposite gradings, $e^{i\tilde{p}_k}$ and $e^{i\hat{p}_l}$, coincide is quite different: Consider the submatrix of $\Omega(z)$ on the subspace of the two associated eigenvectors

$$\Gamma = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array} \right) \quad (2.34)$$

The eigenvalues of this supermatrix Γ are given by

$$\gamma_1 = \frac{bc}{d-a} + a, \quad \gamma_2 = \frac{bc}{d-a} + d, \quad (2.35)$$

where again a, b, c, d are given by analytic functions in z . At $z = z_a^*$, the combination $f = a - d = \gamma_1 - \gamma_2 = \text{str } \Gamma$ in the denominator is zero by definition, $f(z_a^*) = 0$. Generically, we cannot however expect that also the numerator bc vanishes and therefore we find a pole singularity at z_a^*

$$e^{i\tilde{p}_k(z)} = e^{i\tilde{p}_k(z_a^*)} \left(\frac{\alpha_a^*}{z - z_a^*} + 1 + \mathcal{O}(z - z_a^*) \right) = e^{i\hat{p}_l(z)}. \quad (2.36)$$

Note that the residue α_a^* as well as the regular part $e^{i\tilde{p}_k(z_a^*)}$ of $e^{i\tilde{p}_k(z)}$ at $z = z_a^*$ are the same for both eigenvalues.¹⁵ We thus learn that the set of eigenvalues of $\Omega(z)$ depends analytically on z except at a set of points $\{0, \infty, \tilde{z}_a, \hat{z}_a, z_a^*\}$. Let us assume that there are only finitely many singularities of this kind. The cases of an infinite number of singularities can hopefully be viewed as limits of this finite setting. A unique labelling of eigenvalues cannot be achieved globally, because a full circle around one of the square-root singularities \tilde{z}_a, \hat{z}_a will result in an interchange of the two eigenvalues associated to the singularity. Therefore we need to introduce several branch cuts $\tilde{\mathcal{C}}_a$ and $\hat{\mathcal{C}}_a$ in $e^{i\tilde{p}_k(z)}$ and $e^{i\hat{p}_k(z)}$, respectively, which connect the square-root singularities. The functions $\tilde{p}_k(z)$ and $\hat{p}_k(z)$ are therefore analytic except at $\{0, \infty, \tilde{\mathcal{C}}_a, \hat{\mathcal{C}}_a, z_a^*\}$. Alternatively, we could view $e^{i\tilde{p}(z)}$ and $e^{i\hat{p}(z)}$ as one function on suitable four-fold coverings $\tilde{\mathbb{M}}$ and $\hat{\mathbb{M}}$ of $\tilde{\mathbb{C}}$. In that case, the functions $e^{i\tilde{p}(z)}$ and $e^{i\hat{p}(z)}$ are analytic except at $\{0, \infty, \tilde{z}_a, \hat{z}_a, z_a^*\}$. At z_a^* both functions $e^{i\tilde{p}(z)}$ and $e^{i\hat{p}(z)}$ have poles with equal residues and regular parts. Finally, at 0 and ∞ there are essential singularities of the type $e^{i\alpha_0/z^2}, e^{i\alpha_\infty z^2}$.

¹⁵It might be worthwhile to point out that $\alpha_a^* = -bc$ is the product of two Grassmann-odd quantities and thus, in principle, cannot be an ordinary number. It satisfies a nilpotency condition $(\alpha_a^*)^2 = 0$ which, however, does not quite make it trivial. When quantizing the string, these factors give rise to fermionic excitations due to quantum $\hbar \sim 1/L$ effects. This effect can already be seen in the one-loop spin chain for the $\mathcal{N} = 4$ gauge theory [22] where there are no nilpotent objects.

Except for the last two singularities, the functions $e^{i\hat{p}(z)}, e^{i\tilde{p}(z)}$ would satisfy all requirements for algebraic curves. In order to turn the essential singularities at $\{0, \infty\}$ into regular singularities, we take the logarithmic derivative of the eigenvalues. Let us define the matrix $Y(z)$ according to

$$u(z)Y(z)u^{-1}(z) = -iz \frac{\partial}{\partial z} \log(u(z)\Omega(z)u^{-1}(z)), \quad (2.37)$$

where $u(z)$ diagonalizes $\Omega(z)$. In other words, the eigenvalues of $Y(z)$ are the logarithmic derivatives of the eigenvalues of $\Omega(z)$. The corresponding eigenvectors are the same. We can now reduce $Y(z)$ to the following expression

$$Y(z) = \Omega^{-1}(z)(-iz\Omega'(z) + [U(z), \Omega(z)]), \quad U(z) = -izu^{-1}(z)u'(z). \quad (2.38)$$

As $\Omega(z)$ is non-zero and its only singularities are at $\{0, \infty\}$, any further singularities can only originate from $U(z)$. The diagonalization matrix $u(z)$ has square roots and branch cuts. It appears that all the branch points of $u(z)$ are turned into single poles in $U(z)$.¹⁶ Consequently, $U(z)$ has poles at $\{\tilde{z}_a, \hat{z}_a, z_a^*\}$, but all the branch cuts are removed. Therefore $Y(z)$ is single-valued and analytic on the complex plane except at the singularities $\bar{\mathbb{C}} \setminus \{0, \infty, \tilde{z}_a, \hat{z}_a, z_a^*\}$.

Now we can read off the eigenvalues $\tilde{y}(z), \hat{y}(z)$ of $Y(z)$ from its characteristic function $F(y, z)$

$$F(\tilde{y}(z), z) = 0, \quad F(\hat{y}(z), z) = \infty \quad (2.39)$$

with

$$F(y, z) = \frac{\tilde{F}_4(z)}{\hat{F}_4(z)} \text{sdet}(y - Y(z)) = \frac{\tilde{F}(y, z)}{\hat{F}(y, z)}. \quad (2.40)$$

We have included polynomial prefactors $\tilde{F}_4(z), \hat{F}_4(z)$ in the definition of $F = \tilde{F}/\hat{F}$ which clearly do not change the algebraic curve. The purpose of the prefactors is to remove the poles originating from $U(z)$. The roots of these prefactors are thus given by the singularities $\{\tilde{z}_a, z_a^*\}$ or $\{\hat{z}_a, z_a^*\}$, respectively. They enable us to write both \tilde{F} and \hat{F} as polynomials, not only in y (obvious), but also in z .

As $Y(z)$ has only pole-type singularities at $\{0, \infty, \tilde{z}_a, \hat{z}_a, z_a^*\}$, the above equation defines two algebraic curves $\tilde{y}(z)$ and $\hat{y}(z)$ on the Riemann surfaces $\tilde{\mathbb{M}}$ and $\hat{\mathbb{M}}$, respectively. We can even unite the two curves into one curve $y(z) = \{\hat{y}(z) || \tilde{y}(z)\}$ on $\mathbb{M} = \tilde{\mathbb{M}} \cup \hat{\mathbb{M}}$. At the points $\{\tilde{z}_a\}, \{\hat{z}_a\}$, the functions $\tilde{y}(z), \hat{y}(z)$ have inverse square-root singularities. At $\{z_a^*\}$ both functions $\tilde{y}(z), \hat{y}(z)$ have double poles with equal coefficients. Similarly, at $\{0, \infty\}$, there are singularities of the type $-2\alpha_0/z^2, 2\alpha_\infty z^2$. Finally, there are no single poles anywhere, because they would lead to a singular matrix Ω , which cannot happen.

2.4 The Central Element

Consider the local transformation

$$g(\tau, \sigma) \mapsto \xi(\tau, \sigma) g(\tau, \sigma) \quad (2.41)$$

¹⁶This may require a special matrix $u(z)$. The point is that one can redefine $u(z) \mapsto a(z)u(z)$ with any diagonal matrix $a(z)$. This is a possible source of non-analyticity in $U(z)$, which however drops out in $[U(z), \Omega(z)]$.

with ξ a number-valued field which is nowhere zero. Here we would like to demonstrate that this transformation does not have any physical effect. First of all, it changes the current J by $J \mapsto J - \xi^{-1}d\xi$, but note that $\text{str } J = 0$ remains true due to $\text{str } I = 0$. The transformation can now be easily seen to affect only the P -component of J

$$P \mapsto P - \xi^{-1}d\xi. \quad (2.42)$$

In the equations of motion (2.13) the variation drops out when the Lagrange multiplier shifts accordingly

$$\Lambda \mapsto \Lambda - \xi^{-1}*d\xi - i d\zeta - iv d\sigma. \quad (2.43)$$

The additional transformation parameters are the field $\zeta(\tau, \sigma)$ and the constant v . We cannot include v in ζ as $\zeta \rightarrow \zeta + v\sigma$ as ζ would not be periodic. The action is also invariant except for the term proportional to v . This actually leads to a change of the global charges (2.17)

$$S \mapsto S - \frac{\sqrt{\lambda}}{2\pi} \oint d\zeta - v \frac{\sqrt{\lambda}}{2\pi} \oint d\sigma = S - \sqrt{\lambda} v. \quad (2.44)$$

This change of the central element of S is unphysical because the global symmetry is merely $\text{PSU}(2, 2|4)$, not $\text{SU}(2, 2|4)$.

The family of flat connections changes up to a central gauge transformation

$$A(z) \mapsto A(z) - (\tfrac{1}{2}z^2 + \tfrac{1}{2}z^{-2})\xi^{-1}d\xi - (\tfrac{1}{2}z^2 - \tfrac{1}{2}z^{-2})(i d\zeta + iv d\sigma). \quad (2.45)$$

As this is an abelian shift, it completely factorizes from the monodromy and we get

$$\Omega(z) \mapsto \Omega(z) \exp \int_0^{2\pi} d\sigma \left((1 - \tfrac{1}{2}z^2 - \tfrac{1}{2}z^{-2})\xi^{-1}\partial_\sigma\xi - (\tfrac{1}{2}z^2 - \tfrac{1}{2}z^{-2})(i\partial_\sigma\zeta + iv) \right). \quad (2.46)$$

The first term measures the winding number of ξ around 0 when going once around the string. This winding affects both the AdS_5 and S^5 parts of g . However, in the physical setting, the background is a universal cover and windings around the time-circle of AdS_5 are not permitted. Therefore the term involving ξ does not contribute. Also the term involving ζ vanishes because ζ is periodic. We end up with

$$\Omega(z) \mapsto \Omega(z) \exp(-i\pi v(z^2 - z^{-2})). \quad (2.47)$$

The factor is abelian and does not change the eigenvectors. We thus find

$$Y(z) \mapsto Y(z) - 2\pi v(z^2 + z^{-2}). \quad (2.48)$$

This means that we can shift the curve $y(z)$ by a term proportional to $(z^2 + z^{-2})$ as long as the factor of proportionality is the same for all sheets.

2.5 Symmetry

Let us introduce the (antisymmetric) supermatrix

$$C = E_1 - iE_2. \quad (2.49)$$

Then there is another useful way of expressing (2.9)

$$\begin{aligned} H &= \frac{1}{4}J - \frac{1}{4}C J^{\text{ST}} C^{-1} + \frac{1}{4}\eta J \eta - \frac{1}{4}\eta C J^{\text{ST}} C^{-1} \eta, \\ Q_1 &= \frac{1}{4}J - \frac{i}{4}C J^{\text{ST}} C^{-1} - \frac{1}{4}\eta J \eta + \frac{i}{4}\eta C J^{\text{ST}} C^{-1} \eta, \\ P &= \frac{1}{4}J + \frac{1}{4}C J^{\text{ST}} C^{-1} + \frac{1}{4}\eta J \eta + \frac{1}{4}\eta C J^{\text{ST}} C^{-1} \eta, \\ Q_2 &= \frac{1}{4}J + \frac{i}{4}C J^{\text{ST}} C^{-1} - \frac{1}{4}\eta J \eta - \frac{i}{4}\eta C J^{\text{ST}} C^{-1} \eta, \end{aligned} \quad (2.50)$$

where η is the grading matrix (2.4). This form reveals that a conjugation of the four components H, Q_1, P, Q_2 of J with C is equivalent to their supertranspose up to a sign determined by their grading under \mathbb{Z}_4

$$\begin{aligned} C^{-1} H C &= -H^{\text{ST}}, \\ C^{-1} Q_1 C &= -iQ_1^{\text{ST}}, \\ C^{-1} P C &= +P^{\text{ST}}, \\ C^{-1} Q_2 C &= +iQ_2^{\text{ST}}. \end{aligned} \quad (2.51)$$

When we apply this conjugation to the flat connections we obtain

$$\begin{aligned} C^{-1} A(z) C &= -H^{\text{ST}} + \left(\frac{1}{2}z^2 + \frac{1}{2}z^{-2}\right) P^{\text{ST}} + \left(-\frac{1}{2}z^2 + \frac{1}{2}z^{-2}\right) * P^{\text{ST}} - iz^{-1} Q_1^{\text{ST}} + iz Q_2^{\text{ST}} \\ &= -A^{\text{ST}}(-iz). \end{aligned} \quad (2.52)$$

This, in turn, implies a symmetry relation for the monodromy $C^{-1} \Omega(z) C = \Omega^{-\text{ST}}(-iz)$. The inverse is due to the overall sign in (2.52) and the transpose puts the Wilson loop in the original path ordering. In other words¹⁷

$$\Omega(iz) = C \Omega^{-\text{ST}}(z) C^{-1} \quad (2.53)$$

is related to $\Omega(z)$ by conjugation, inversion and supertranspose. This translates to the following symmetry of $Y(z)$ and $U(z)$

$$Y(iz) = -C Y^{\text{ST}}(z) C^{-1}, \quad U(iz) = -C U^{\text{ST}}(z) C^{-1}. \quad (2.54)$$

In particular, the characteristic function has the symmetry

$$F(y, iz) = \frac{\tilde{F}_4(iz)}{\hat{F}_4(iz)} \text{sdet}(y - Y(iz)) = \frac{\hat{F}_4(z)}{\tilde{F}_4(z)} \text{sdet}(y + Y(z)) = F(-y, z). \quad (2.55)$$

It therefore depends analytically only on the combinations z^4, yz^2, y^2 . In other words, $y(iz) = -y(z)$ and consequently $p(iz) = -p(z) + 2\pi\mathbb{Z}$ with some permutation of the sheets.

¹⁷The contribution from $\Omega_0^{-1}(1) = h(0)$ can be seen to cancel, because $h \in \text{Sp}(4, \mathbb{R}) \times \text{Sp}(4, \mathbb{R})$.

To determine the permutation, let us consider the action on the diagonalized matrix

$$Y_{\text{diag}}(iz) = -C_{\text{diag}}(z)Y_{\text{diag}}^{\text{ST}}(z)C_{\text{diag}}^{-1}(z) \quad \text{with} \quad C_{\text{diag}}(z) = u(iz)C u^{\text{ST}}(z). \quad (2.56)$$

As both $Y_{\text{diag}}(iz)$ and $Y_{\text{diag}}^{\text{ST}}(iz) = Y_{\text{diag}}(iz)$ are diagonal, $C_{\text{diag}}(z)$ must be a permutation matrix and thus constant (up to branch cuts). In particular, we should investigate the fixed points of $z \mapsto iz$; these are the singular points $\{0, \infty\}$. At these points, $C_{\text{diag}}(z)$ as defined in (2.56) must approach an antisymmetric matrix related to C . As it is constant, it must always be an antisymmetric permutation matrix which acts non-trivially with period 2. We therefore find that the eigenvalues obey the symmetry

$$\tilde{y}_k(iz) = -\tilde{y}_{k'}(z), \quad \hat{y}_k(iz) = -\hat{y}_{k'}(z) \quad (2.57)$$

where we are free to choose the following permutation of sheets

$$k' = (2, 1, 4, 3) \quad \text{for} \quad k = (1, 2, 3, 4). \quad (2.58)$$

For the quasi-momentum we find

$$\tilde{p}_k(iz) = 2\pi m \varepsilon_k - \tilde{p}_{k'}(z), \quad \hat{p}_k(iz) = -\hat{p}_{k'}(z). \quad (2.59)$$

Here we have introduced

$$\varepsilon_k = (+1, +1, -1, -1) \quad \text{for} \quad k = (1, 2, 3, 4). \quad (2.60)$$

The constant shift $2\pi m$ in $\tilde{p}_k(iz)$ is related to winding around S^5 . It must be absent for the AdS_5 counterpart $\hat{p}_k(iz)$ because there cannot be windings in the time direction.

Finally, we see that y must depend analytically on z^2 . We can thus introduce the variable x defined by

$$x = \frac{1 + z^2}{1 - z^2}, \quad z^2 = \frac{x - 1}{x + 1}, \quad (2.61)$$

which is precisely the variable commonly used for bosonic sigma models as in [17, 29]. The points associated to local and global charges, discussed in the following subsections, and the symmetry are related as follows, see also Fig. 2

$$\begin{aligned} x = \infty & \Leftrightarrow z = \pm 1, \\ x = 0 & \Leftrightarrow z = \pm i, \\ x = +1 & \Leftrightarrow z = 0, \\ x = -1 & \Leftrightarrow z = \infty, \\ x \mapsto 1/x & \Leftrightarrow z \mapsto iz. \end{aligned} \quad (2.62)$$

Note the relation of differentials

$$\frac{dx}{1 - 1/x^2} = \frac{dz}{z} = d\kappa, \quad (2.63)$$

where $\kappa = \log z$ is the spectral parameter used in [34].

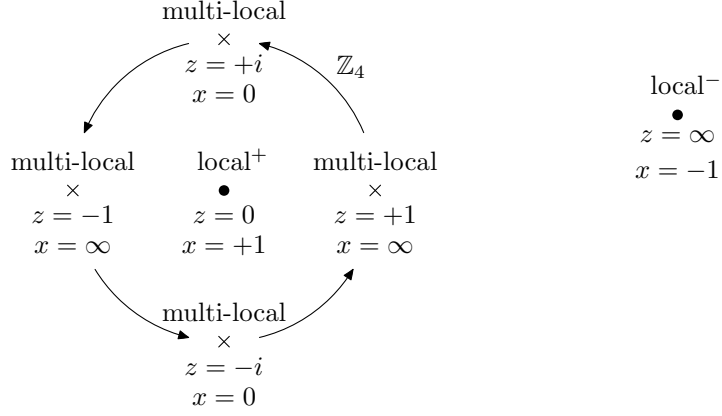


Figure 2: Special points of the quasi-momenta. The expansion around $z = 0, \infty$ yields one sequence of local charges each, see Sec. 2.6. At $z = \pm 1, \pm i$ one finds the Noether charges, discussed in Sec. 2.8, and multi-local charges. All other points are related to non-local charges.

2.6 Local Charges

At the points $z = 0, \infty$ the expansion of the Lax connection is singular

$$A(\epsilon^{\pm 1}) = \frac{1}{2}\epsilon^{-2}(P \pm *P \mp \Lambda) + \epsilon^{-1}Q_{1,2} + H + \epsilon Q_{2,1} + \frac{1}{2}\epsilon^2(P \mp *P \pm \Lambda). \quad (2.64)$$

The expansion of the quasi-momentum $p(z)$ at these points is thus related to local charges. As was shown in, e.g., [29], in the absence of the fermionic contributions $Q_{1,2}$, the leading coefficient of $p(z)$ in ϵ is directly related to eigenvalues of the leading contribution to A_σ . Let us repeat the argument for the point $z = 0$. Consider the transformed connection $\bar{A}(z)$ in the σ -direction given by

$$\partial_\sigma - \bar{A}(z) = T(z)(\partial_\sigma - A_\sigma(z))T^{-1}(z). \quad (2.65)$$

Here $T(z)$ and $A(z)$ are given by their expansion in z

$$T(z) = \sum_{r=0}^{\infty} z^r T_r, \quad \bar{A}(z) = \sum_{r=-2}^{\infty} z^r \bar{A}_r. \quad (2.66)$$

We demand that T_0 diagonalizes the leading term

$$\bar{A}_{-2} = \frac{1}{2}T_0 P_+ T_0^{-1} + \frac{1}{2}\Lambda_\sigma = \text{diag}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4 || \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4). \quad (2.67)$$

Since P satisfies $CP^{\text{ST}}C^{-1} = P$, c.f. (2.51), its eigenvalues must be doubly degenerate, $\tilde{\alpha}_1 = \tilde{\alpha}_2$, $\tilde{\alpha}_3 = \tilde{\alpha}_4$, $\hat{\alpha}_1 = \hat{\alpha}_2$, $\hat{\alpha}_3 = \hat{\alpha}_4$. Furthermore, P_+ satisfies the Virasoro constraint $\text{str } P_+^2 = 0$. This requires $\tilde{\alpha}_1 = \hat{\alpha}_1$, $\tilde{\alpha}_3 = \hat{\alpha}_3$. The abelian shift by Λ_σ is compatible with this construction and we find [31]

$$\bar{A}_{-2} = \text{diag}(\alpha, \alpha, \beta, \beta || \alpha, \alpha, \beta, \beta) = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}. \quad (2.68)$$

Here we have introduced a $(2|2) \times (2|2)$ block decomposition of the $(4|4) \times (4|4)$ supermatrix, i.e. each block is a supermatrix. As the eigenvalues α and β are (generically) distinct, we can use the $T_r(z)$ to bring $\bar{A}_{r-2}(z)$ to a block-diagonal form

$$\bar{A}_r = \begin{pmatrix} a_r & 0 \\ 0 & b_r \end{pmatrix} \quad \text{or} \quad \bar{A}(z) = \begin{pmatrix} a(z) & 0 \\ 0 & b(z) \end{pmatrix}. \quad (2.69)$$

When this is done order by order in perturbation theory, the resulting a_r and b_r are local combinations of the fields. However, the diagonalization of the Lax connection is not yet complete and a complete diagonalization will lead non-local results. Still we can obtain local charges: Although the open Wilson loop Ω is in general non-local, its superdeterminant is the exponential of a local charge. Here $\text{sdet } \Omega = 1$ is trivial, but we can consider only one block of $T(2\pi)\Omega T(0)^{-1}$

$$\omega(z) = \left(\text{P exp} \int_0^{2\pi} a(1) \right)^{-1} \left(\text{P exp} \int_0^{2\pi} a(z) \right). \quad (2.70)$$

Then $\text{sdet } \omega(z) = \exp iq(z)$ with

$$q(z) = -i \int_0^{2\pi} d\sigma \left(\text{str } a(z) - \text{str } a(1) \right). \quad (2.71)$$

The expression for the other block involving $b(z)$ is in fact equivalent due to $\text{str } a + \text{str } b = \text{str } \bar{A} = 0$. The expansion of $q(z)$ into q_r gives a sequence of local charges. The term q_{-2} vanishes because a_{-2} is proportional to the identity. We can also perform a similar construction around $z = \infty$ leading to similar charges and thus we have found two infinite sequences of local charges. Let us express $q(z)$ through the quasi-momentum $p(z)$. As $\exp iq$ is the superdeterminant of the block ω of $T(2\pi)\Omega T(0)^{-1}$ we can also write $q(z)$ as a sum over a half of the quasi-momenta

$$q(z) = \tilde{p}_1(z) + \tilde{p}_2(z) - \hat{p}_1(z) - \hat{p}_2(z). \quad (2.72)$$

Using (2.60) we write the generator of local charges in the concise form

$$q(z) = \sum_{k=1}^4 \varepsilon_k \left(\frac{1}{2} \tilde{p}_k(z) - \frac{1}{2} \hat{p}_k(z) \right). \quad (2.73)$$

Expanded around $z = 0, \infty$ it yields the conserved local charges. In App. C we will construct the first of these charges. Note that besides the local charges there is a larger set of conserved non-local charges.

2.7 Singularities

We would like to understand the singular behavior of the quasi-momentum $p(z)$ at $z = 0$ better. In the bosonic case we would be finished after the semi-diagonalization of the previous section because all singular terms have been diagonalized and can be integrated up. In the supersymmetric case, the remaining singular term a_{-1} is not diagonal and

might lead to further singularities at $z = 0$. Here we will show that this does not happen. The difficulty of the proof is that any attempt to diagonalize further would lead to non-local terms.

We shall start with one block $a(z)$ of the semi-diagonalized connection $\bar{A}(z)$. Let us investigate the logarithm of the monodromy $\omega(z)$ and expand near $z = 0$ ¹⁸

$$\log \omega(z) = \int_0^{2\pi} d\sigma a(\sigma, z) + \int_0^{2\pi} d\sigma \int_0^\sigma d\sigma' \frac{1}{2} [a(\sigma, z), a(\sigma', z)] + \dots \quad (2.74)$$

The further terms involve nested commutators of $a(z)$ at various points σ . The term $a_{-2} = \alpha I$ is abelian and thus contributes only to the first term. This is not necessarily the most singular term, as a_{-1} may appear many times within the nested commutators. To resolve this problem we note that the full Lax connection obeys the \mathbb{Z}_4 -symmetry relation (2.51). This reduces to a similar relation for the block $a(z)$

$$c a^{\text{ST}}(z) c^{-1} = -a(iz) \quad \text{or} \quad c a_r^{\text{ST}} c^{-1} = i^{2+r} a_r. \quad (2.75)$$

where $c = e_1 - ie_2$ with $e_{1,2}$ as in (2.1), but with e being a 2×2 instead of a 4×4 antisymmetric matrix. This means that a_r has \mathbb{Z}_4 -grading r . Note that the grading is obeyed by commutators, i.e. when x and y have gradings r and s , respectively, the commutator $[x, y]$ has grading $r + s$. Now consider two $(2|2) \times (2|2)$ supermatrices x, y of grading -1 . Then it can be shown (explicitly) that their commutator $[x, y]$ is proportional to the identity matrix I . It therefore drops out of any further commutators and nested commutators can never produce terms of grading less than -2 . Furthermore, all terms of grading -2 are proportional to the identity. The grading coincides with the power of z and we find

$$\log \omega(z) = d_{-2} z^{-2} I + d_{-1} z^{-1} + \mathcal{O}(z^0) \quad (2.76)$$

with d_{-2} a number and d_{-1} a matrix of grading -1 . To finally diagonalize $\log \omega(z)$ we first use a matrix $\exp(t_{-1} z^{-1})$ which, using the Baker-Campbell-Hausdorff identity and for the same reasons as above, removes the term d_{-1} without lifting the degeneracy of double poles or creating even higher poles. Afterwards $\omega(z)$ can be diagonalized perturbatively. Of course, all of the above holds true for the other block. We assemble the two blocks and find for the quasi-momentum

$$\tilde{p}_k(z) \sim \hat{p}_k(z) \sim (\alpha_0 + \varepsilon_k \beta_0) z^{-2} + \mathcal{O}(z^{-1}) \quad (2.77)$$

with some coefficients α_0, β_0 not directly related to α, β . We have thus proved that the structure of residues found in [31] is not affected by the fermions. Note that this distribution on the p_k is compatible with the permutation of sheets in Sec. 2.5. Similarly, at $z = \infty$ the expansion of the quasi-momenta is given by

$$\tilde{p}_k(z) \sim \hat{p}_k(z) \sim (\alpha_\infty + \varepsilon_k \beta_\infty) z^2 + \mathcal{O}(z). \quad (2.78)$$

¹⁸For convenience we omit contributions from the second term in ω ; they do not change the principal result.

2.8 Global Charges

At $z = 1$ the expansion of the Lax connection

$$A(1 + \epsilon) = J - 2\epsilon * K + \mathcal{O}(\epsilon^2) \quad (2.79)$$

is related to the $\mathfrak{psu}(2, 2|4)$ Noether current. The expansion of the monodromy yields

$$\Omega(1 + \epsilon) = I - 2\epsilon \int_0^{2\pi} d\sigma \left(\text{P exp} \int_0^\sigma d\sigma' J_\sigma(\sigma') \right)^{-1} K_\tau(\sigma) \left(\text{P exp} \int_0^\sigma d\sigma' J_\sigma(\sigma') \right) + \mathcal{O}(\epsilon^2) \quad (2.80)$$

which equals

$$\Omega(1 + \epsilon) = I - \epsilon \frac{4\pi S}{\sqrt{\lambda}} + \mathcal{O}(\epsilon^2) \quad (2.81)$$

by means of (2.17). Not only $z = +1$, but also $z = -1$ and $z = \pm i$ are related to the global charges, as can be seen from the symmetry discussed in Sec. 2.5. The higher orders in the expansion yield multi-local charges. These are the Yangian generators discussed in [34, 37, 42, 38].

The expansion of the quasi-momenta $\tilde{p}_k(z)$ associated to S^5 at $z = 1$ is [29]

$$\begin{aligned} \tilde{p}_1(1 + \epsilon) &= -\epsilon \frac{4\pi}{\sqrt{\lambda}} \left(+\frac{3}{4}\tilde{r}_1 + \frac{1}{2}\tilde{r}_2 + \frac{1}{4}\tilde{r}_3 + \frac{1}{4}r^* \right) + \dots, \\ \tilde{p}_2(1 + \epsilon) &= -\epsilon \frac{4\pi}{\sqrt{\lambda}} \left(-\frac{1}{4}\tilde{r}_1 + \frac{1}{2}\tilde{r}_2 + \frac{1}{4}\tilde{r}_3 + \frac{1}{4}r^* \right) + \dots, \\ \tilde{p}_3(1 + \epsilon) &= -\epsilon \frac{4\pi}{\sqrt{\lambda}} \left(-\frac{1}{4}\tilde{r}_1 - \frac{1}{2}\tilde{r}_2 + \frac{1}{4}\tilde{r}_3 + \frac{1}{4}r^* \right) + \dots, \\ \tilde{p}_4(1 + \epsilon) &= -\epsilon \frac{4\pi}{\sqrt{\lambda}} \left(-\frac{1}{4}\tilde{r}_1 - \frac{1}{2}\tilde{r}_2 - \frac{3}{4}\tilde{r}_3 + \frac{1}{4}r^* \right) + \dots. \end{aligned} \quad (2.82)$$

Here, $[\tilde{r}_1, \tilde{r}_2, \tilde{r}_3]$ are the the Dynkin labels of $\text{SU}(4)$ related to the spins of $\text{SO}(6)$ by $\tilde{r}_1 = J_2 - J_3$, $\tilde{r}_2 = J_1 - J_2$, $\tilde{r}_3 = J_2 + J_3$. The label r^* is an unphysical label related to the $\text{U}(1)$ hypercharge. It transforms under the transformation described in Sec. 2.4 as $r^* \mapsto r^* + v\sqrt{\lambda}$. Similarly, the expansion for $\hat{p}_k(z)$ associated to AdS_5 reads [31]

$$\begin{aligned} \hat{p}_1(1 + \epsilon) &= \epsilon \frac{4\pi}{\sqrt{\lambda}} \left(+\frac{3}{4}\hat{r}_1 + \frac{1}{2}\hat{r}_2 + \frac{1}{4}\hat{r}_3 - \frac{1}{4}r^* \right) + \dots, \\ \hat{p}_2(1 + \epsilon) &= \epsilon \frac{4\pi}{\sqrt{\lambda}} \left(-\frac{1}{4}\hat{r}_1 + \frac{1}{2}\hat{r}_2 + \frac{1}{4}\hat{r}_3 - \frac{1}{4}r^* \right) + \dots, \\ \hat{p}_3(1 + \epsilon) &= \epsilon \frac{4\pi}{\sqrt{\lambda}} \left(-\frac{1}{4}\hat{r}_1 - \frac{1}{2}\hat{r}_2 + \frac{1}{4}\hat{r}_3 - \frac{1}{4}r^* \right) + \dots, \\ \hat{p}_4(1 + \epsilon) &= \epsilon \frac{4\pi}{\sqrt{\lambda}} \left(-\frac{1}{4}\hat{r}_1 - \frac{1}{2}\hat{r}_2 - \frac{3}{4}\hat{r}_3 - \frac{1}{4}r^* \right) + \dots. \end{aligned} \quad (2.83)$$

The Dynkin labels $[\hat{r}_1, \hat{r}_2, \hat{r}_3]$ of $\text{SU}(2, 2)$ are related to the spins of $\text{SO}(2, 4)$ by $\hat{r}_1 = S_1 - S_2$, $\hat{r}_2 = -E - S_1$, $\hat{r}_3 = S_1 + S_2$.

2.9 Bosonic $AdS_5 \times S^5$, $\mathbb{R} \times S^5$ and $AdS_5 \times S^1$ Sectors

The restriction to the classical bosonic string on $AdS_5 \times S^5$ [31], $\mathbb{R} \times S^5$ [29] and $AdS_5 \times S^1$ [30] is straight-forward: First of all we remove all possible fermionic poles. This implies $K^* = 0$ but we can also set $r^* = B = 0$ and obtain the bosonic string on $AdS_5 \times S^5$. Then the expansion at $z = 1$ (2.82, 2.83) as well as the structure of poles at $z = 0, \infty$, c.f. Sec. 2.7, agrees with [31] under the change of spectral parameter (2.61).

In the next step we either reduce AdS_5 to \mathbb{R} or S^5 to S^1 . The isometry groups of both factors \mathbb{R} and S^1 are abelian. For the monodromy corresponding to this factor we can therefore remove the path ordering

$$\Omega(z) = \left(\text{P exp} \oint A(1) \right)^{-1} \left(\text{P exp} \oint A(z) \right) = \exp \oint (A(z) - A(1)), \quad (2.84)$$

We now substitute $A(z)$ from (2.24) with $H = Q_1 = Q_2 = 0$, $P = -g^{-1}dg$ and solve $\Omega(z) = e^{ip(z)}$ for the quasi-momentum

$$p(z) = i(1 - \frac{1}{2}z^2 + \frac{1}{2}z^{-2}) \oint g^{-1}dg - i(-\frac{1}{2}z^2 + \frac{1}{2}z^{-2}) \oint g^{-1} * dg. \quad (2.85)$$

The first integral represents the winding number m , it must vanish for \mathbb{R} and can be non-trivial for S^1 . The second integral represents the global charge, it is proportional to the energy E for \mathbb{R} and to the spin J for S^1 . By comparing to (2.83) we find that in the case of $\mathbb{R} \times S^5$ the full quasi-momentum for AdS_5 is given by

$$\hat{p}_k(z) = \varepsilon_k \frac{\pi E}{\sqrt{\lambda}} \left(-\frac{1}{2}z^2 + \frac{1}{2}z^{-2} \right). \quad (2.86)$$

When the residues at $z = 0, \infty$ are matched between \tilde{p}_k and \hat{p}_k we find perfect agreement with [29]. Equivalently in the case of $AdS_5 \times S^1$ the full quasi-momentum for S^5 is obtained by comparing to (2.82)¹⁹

$$\tilde{p}_k(z) = \varepsilon_k \frac{\pi J}{\sqrt{\lambda}} \left(-\frac{1}{2}z^2 + \frac{1}{2}z^{-2} \right) + \varepsilon_k \pi m \left(1 - \frac{1}{2}z^2 - \frac{1}{2}z^{-2} \right). \quad (2.87)$$

Again, after matching the residues, this is in agreement with [30].

3 Moduli of the Curve

In this section we investigate the moduli space of admissible curves. Admissible curves are algebraic curves which satisfy all the properties derived in the previous section and which can thus arise from a classical string configuration on $AdS_5 \times S^5$. For a fixed degree of complexity of the solution, which manifests as the genus of the curve, we count the number of degrees of freedom for admissible curves. Although it is not obvious that all admissible curves indeed represent string solutions (in other words that we have identified

¹⁹The integral of $g^{-1}dg$ yields odd multiples of $i\pi$ when $g(2\pi) = -g(0)$, which is an allowed case.

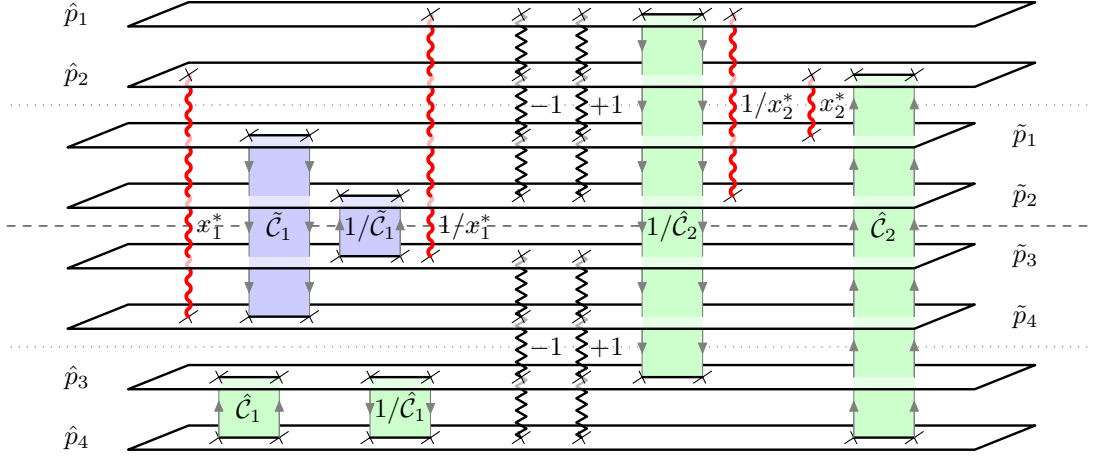


Figure 3: Some configuration of cuts and poles for the sigma model. Cuts $\tilde{\mathcal{C}}_a$ between the sheets \tilde{p}_k correspond to S^5 excitations and likewise cuts $\hat{\mathcal{C}}_a$ between the sheets \hat{p}_k correspond to AdS^5 excitations. Poles x_a^* on sheets \tilde{p}_k and \hat{p}_l correspond to fermionic excitations. The dashed line in the middle is related to physical excitations, cuts and poles which cross it contribute to the total momentum, energy shift and local charges.

all relevant properties of admissible curves) we see that this number agrees with strings in flat space. We take this as evidence that our classification of string solutions in terms of admissible curves is complete. We finally identify the discrete parameters and continuous moduli with certain cycles on the curve and interpret them. For the comparison to gauge theory we investigate the Frolov-Tseytlin limit of the algebraic curve corresponding to a loop expansion in gauge theory.

3.1 Properties

Let us collect the analytic properties of the quasi-momentum

$$p(x) = \{\tilde{p}_1(x), \tilde{p}_2(x), \tilde{p}_3(x), \tilde{p}_4(x) \parallel \hat{p}_1(x), \hat{p}_2(x), \hat{p}_3(x), \hat{p}_4(x)\}, \quad (3.1)$$

see Fig. 3 for an illustration. All sheet functions $\tilde{p}_k(x)$ and $\hat{p}_l(x)$ are analytic almost everywhere. The singularities are as follows:

- At $x = \pm 1$ there are single poles, c.f. Sec. 2.6. The four sheets $\tilde{p}_{1,2}(x), \hat{p}_{1,2}(x)$ all have equal residues; the same holds for the remaining four sheets $\tilde{p}_{3,4}(x), \hat{p}_{3,4}(x)$.
- Bosonic degrees of freedom are represented by branch cuts $\{\tilde{\mathcal{C}}_a\}$, $a = 1, \dots, 2\tilde{A}$ and $\{\hat{\mathcal{C}}_a\}$, $a = 1, \dots, 2\hat{A}$. The cut $\tilde{\mathcal{C}}_a$ connects the sheets \tilde{k}_a and \tilde{l}_a of $\tilde{p}'(x)$. Equivalently, $\hat{\mathcal{C}}_a$ connects the sheets \hat{k}_a and \hat{l}_a of $\hat{p}'(x)$. At both ends of the branch cut, \tilde{x}_a^\pm or \hat{x}_a^\pm , there is a square-root singularity on both sheets.
- Fermionic degrees of freedom are represented by poles at $\{x_a^*\}$, $a = 1, \dots, 2A^*$. The pole x_a^* exists on the sheets k_a^* of $\tilde{p}(x)$ and l_a^* of $\hat{p}(x)$ with equal residue.

Further properties are:

- For definiteness, we assume the quasi-momentum to approach zero at $x = \infty$ on all sheets, c.f. Sec. 2.8

$$\tilde{p}(x) = \mathcal{O}(1/x), \quad \hat{p}(x) = \mathcal{O}(1/x). \quad (3.2)$$

- The quasi-momentum obeys the symmetry $x \mapsto 1/x$, see Sec. 2.5, as follows

$$\tilde{p}_k(1/x) = -\tilde{p}_{k'}(x) + 2\pi m \varepsilon_k, \quad \hat{p}_k(1/x) = -\hat{p}_{k'}(x). \quad (3.3)$$

We use the permutation k' of k and a sign ε_k for each sheet $k = (1, 2, 3, 4)$ as defined in (2.58, 2.60)

$$k' = (2, 1, 4, 3), \quad \varepsilon_k = (+1, +1, -1, -1). \quad (3.4)$$

The branch cuts and poles must respect the symmetry. We therefore consider the cut $\tilde{\mathcal{C}}_{\tilde{A}+a} = 1/\tilde{\mathcal{C}}_a$ to be the image of $\tilde{\mathcal{C}}_a$. The independent cuts are thus labelled by $a = 1, \dots, \tilde{A}$. Similarly for AdS_5 -cuts $\hat{\mathcal{C}}_a$ and fermionic poles x_a^* ²⁰

$$\tilde{\mathcal{C}}_{\tilde{A}+a} = 1/\tilde{\mathcal{C}}_a, \quad \hat{\mathcal{C}}_{\tilde{A}+a} = 1/\hat{\mathcal{C}}_a, \quad x_{\tilde{A}^*+a}^* = 1/x_a^*. \quad (3.5)$$

Note that there is an arbitrariness of which cuts are considered fundamental and which are their images under the symmetry. E.g. we might replace $\tilde{\mathcal{C}}_a$ by $1/\tilde{\mathcal{C}}_a$ which effectively interchanges $\tilde{\mathcal{C}}_a$ and $\tilde{\mathcal{C}}_{\tilde{A}+a}$ without changing the curve.

- The unimodularity condition (2.29) together with (3.2) translates to

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 = \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4. \quad (3.6)$$

- A common shift of all sheets

$$\tilde{p}(x) \mapsto \tilde{p}(x) - \frac{4\pi v}{1 - 1/x^2}, \quad \hat{p}(x) \mapsto \hat{p}(x) - \frac{4\pi v}{1 - 1/x^2} \quad (3.7)$$

is considered unphysical, c.f. Sec. 2.4.

For the cuts and poles we define several cycles and periods, c.f. Fig. 4:

- We define the cycles $\tilde{\mathcal{A}}_a, \hat{\mathcal{A}}_a$ which surround the cuts $\tilde{\mathcal{C}}_a, \hat{\mathcal{C}}_a$, respectively. The cuts, which connect the branch points $\{\tilde{x}_a^\pm\}, \{\hat{x}_a^\pm\}$, have been arranged in such a way that

$$\oint_{\tilde{\mathcal{A}}_a} d\tilde{p} = 0, \quad \oint_{\hat{\mathcal{A}}_a} d\hat{p} = 0. \quad (3.8)$$

This can be achieved by a reorganization of cuts which corresponds to a $\text{Sp}(2\tilde{A}, \mathbb{Z})$ or $\text{Sp}(2\hat{A}, \mathbb{Z})$ transformation, respectively [17].

²⁰Within sums a self-symmetric cut will be counted with weight 1/2.

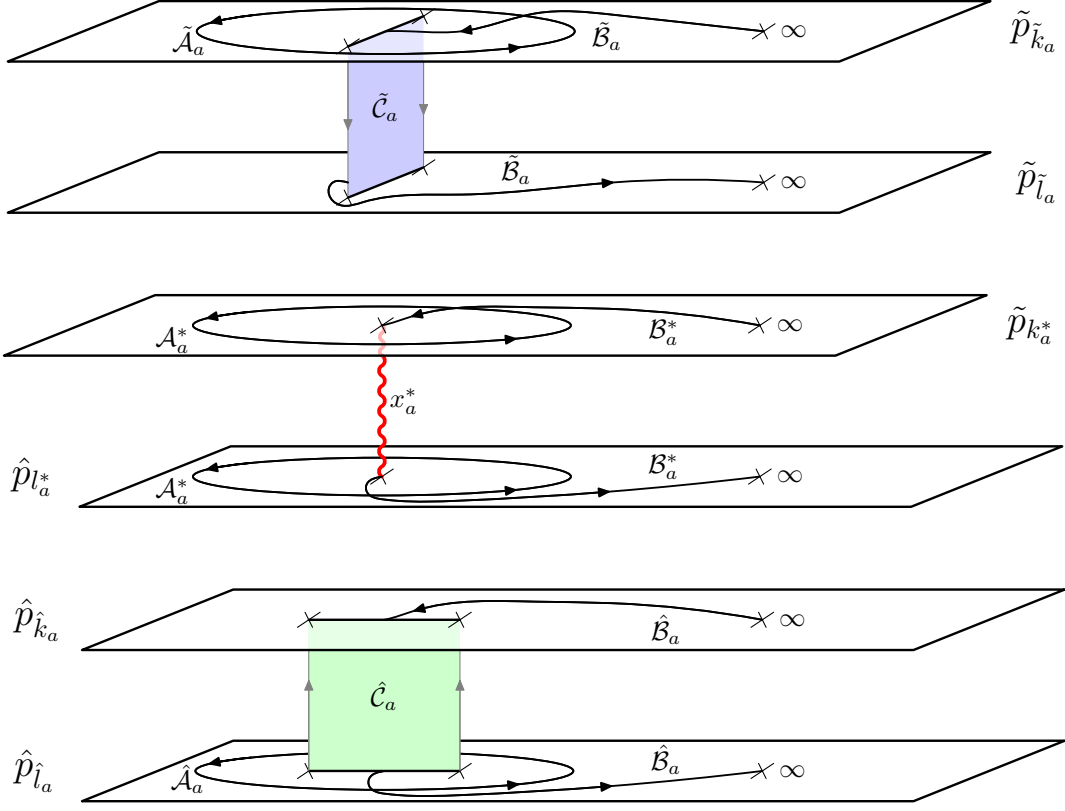


Figure 4: Cycles for S^5 -cuts (top), fermionic poles (middle) and AdS_5 -cuts (bottom). Generically, S^5 -cuts are along aligned in the imaginary direction while AdS_5 -cuts are along the real axis.

- We define the cycle \mathcal{A}_a^* which surrounds the fermionic pole x_a^* . There are no logarithmic singularities at x_a^*

$$\oint_{\mathcal{A}_a^*} d\tilde{p} = \oint_{\mathcal{A}_a^*} d\hat{p} = 0. \quad (3.9)$$

At the singular points $x = \pm 1$ there are no logarithmic singularities either

$$\oint_{\pm 1} d\tilde{p}_k = \oint_{\pm 1} d\hat{p}_k = 0. \quad (3.10)$$

- We define periods $\tilde{\mathcal{B}}_a, \hat{\mathcal{B}}_a$ which connect $x = \infty$ on sheet \tilde{k}_a, \hat{k}_a to $x = \infty$ on sheet \tilde{l}_a, \hat{l}_a through the cuts $\tilde{\mathcal{C}}_a, \hat{\mathcal{C}}_a$, respectively, see Fig. 4. These must be integral

$$\int_{\tilde{\mathcal{B}}_a} d\tilde{p} = 2\pi\tilde{n}_a, \quad \int_{\hat{\mathcal{B}}_a} d\hat{p} = 2\pi\hat{n}_a, \quad (3.11)$$

because the monodromy at both ends of the B-period is trivial, $\Omega(\infty) = I$. Together with the asymptotic behavior (3.2) and single-valuedness (3.8,3.9,3.10) this implies that $\tilde{p}(x), \hat{p}(x)$ must jump by $2\pi\tilde{n}_a, 2\pi\hat{n}_a$ when passing through the cut $\tilde{\mathcal{C}}_a, \hat{\mathcal{C}}_a$, respec-

tively. This is written as the equivalent condition

$$\begin{aligned}\check{p}_{\tilde{l}_a}(x) - \check{p}_{\tilde{k}_a}(x) &= 2\pi\tilde{n}_a \quad \text{for } x \in \tilde{\mathcal{C}}_a, \\ \check{p}_{\hat{l}_a}(x) - \check{p}_{\hat{k}_a}(x) &= 2\pi\hat{n}_a \quad \text{for } x \in \hat{\mathcal{C}}_a.\end{aligned}\tag{3.12}$$

- The period \mathcal{B}_a^* for a fermionic pole connects $x = \infty$ to $x = x_a^*$ on sheet k_a^* of $\tilde{p}(x)$. It then continues from $x = x_a^*$ to $x = \infty$ on sheet l_a^* of $\hat{p}(x)$. As fermionic singularities arise for coinciding eigenvalues, the regular parts of $\tilde{p}(x)$ and $\hat{p}(x)$ must be equal modulo a shift by $2\pi n_a^*$

$$\check{p}_{\hat{l}_a}(x_a^*) - \check{p}_{\tilde{k}_a}(x_a^*) = 2\pi n_a^*.\tag{3.13}$$

Expressed as a B-period this yields

$$\oint_{\mathcal{B}_a^*} dp = 2\pi n_a^*.\tag{3.14}$$

- In addition to $\Omega(\infty) = I$ we also have $\Omega(0) = I$. This means that a period connecting $x = 0$ with $x = \infty$ must be a multiple of 2π . In fact, the symmetry (3.3) enforces

$$\tilde{p}_{1,2}(0) = -\tilde{p}_{3,4}(0) = \int_{\infty}^0 d\tilde{p}_{1,2} = -\int_{\infty}^0 d\tilde{p}_{3,4} = 2\pi m, \quad \hat{p}_k(0) = \int_{\infty}^0 d\hat{p}_k = 0.\tag{3.15}$$

The integral for the AdS_5 -part must vanish, because there cannot be windings on the time circle of AdS_5 [28]. In fact, for physical applications one needs to consider the universal covering of AdS_5 where time circle has been decompactified.

- When no confusion arises, we may use a unified notation \mathcal{A}_a and \mathcal{B}_a with $a = 1, \dots, 2A$ for cuts and poles, $\tilde{\mathcal{A}}_a, \hat{\mathcal{A}}_a, \mathcal{A}_a^*$ and $\tilde{\mathcal{B}}_a, \hat{\mathcal{B}}_a, \mathcal{B}_a^*$. The total number of cuts and poles is $A = \tilde{A} + \hat{A} + A^*$. In this case we label the sheets p_k by $k = 1, \dots, 8$ according to

$$p_{1,2} = \hat{p}_{1,2}, \quad p_{3,4,5,6} = \tilde{p}_{1,2,3,4}, \quad p_{7,8} = \hat{p}_{3,4}.\tag{3.16}$$

This ordering leads to the configuration of sheets as depicted in Fig. 3. Some details of this representation are discussed in App. D. It makes physical excitations and the comparison to gauge theory more transparent.

3.2 Ansatz

The characteristic function of our algebraic curve is rational

$$F(y, x) = \frac{\tilde{F}(y, x)}{\hat{F}(y, x)} = \frac{\tilde{F}_4(x)y^4 + \tilde{F}_3(x)y^3 + \tilde{F}_2(x)y^2 + \tilde{F}_1(x)y + \tilde{F}_0(x)}{\hat{F}_4(x)y^4 + \hat{F}_3(x)y^3 + \hat{F}_2(x)y^2 + \hat{F}_1(x)y + \hat{F}_0(x)},\tag{3.17}$$

with $\tilde{F}_k(x), \hat{F}_k(x)$ polynomials in x . The curve $y(x) = \{\tilde{y}(x) || \hat{y}(x)\}$ obeys the algebraic equation

$$\tilde{F}(\tilde{y}(x), x) = 0, \quad \hat{F}(\hat{y}(x), x) = 0.\tag{3.18}$$

We define the curve $y(x)$ with a different prefactor as compared to the previous section as

$$y(x) = (x - 1/x)^2 x p'(x).\tag{3.19}$$

This definition removes the poles at $x = \pm 1$ [29].

Branch Points and Fermionic Poles. Bosonic branch points \tilde{x}_a^\pm of the S^5 part manifest themselves as inverse square roots in $\tilde{y}(x)$. An asymptotic analysis shows that they are obtained when

$$\tilde{F}_4(\tilde{x}_a^\pm) = \tilde{F}_3(\tilde{x}_a^\pm) = 0 \quad \text{while} \quad \tilde{F}_4'(\tilde{x}_a^\pm) \neq 0 \neq \tilde{F}_3'(\tilde{x}_a^\pm). \quad (3.20)$$

Similarly for branch points \hat{x}_a^\pm in the AdS_5 part

$$\hat{F}_4(\hat{x}_a^\pm) = \hat{F}_3(\hat{x}_a^\pm) = 0 \quad \text{while} \quad \hat{F}_4'(\hat{x}_a^\pm) \neq 0 \neq \hat{F}_3'(\hat{x}_a^\pm). \quad (3.21)$$

Fermionic singularities x_a^* manifest themselves as double poles in $y(x)$. A double pole is achieved by

$$\tilde{F}_4(x_a^*) = \tilde{F}_4'(x_a^*) = \hat{F}_4(x_a^*) = \hat{F}_4'(x_a^*) = 0 \quad \text{while} \quad \tilde{F}_3(x_a^*) \neq 0 \neq \hat{F}_3(x_a^*). \quad (3.22)$$

The behavior of $F_{2,1,0}$ is generic at these points. Here we see that a non-zero F_3 , unlike in [29], is required due to fermions. All these singularities are encoded in $F_4(x)$ as

$$\begin{aligned} \tilde{F}_4(x) &= x^4 \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^+) \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^-) \prod_{a=1}^{2A^*} (x - x_a^*)^2, \\ \hat{F}_4(x) &= x^4 \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^+) \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^-) \prod_{a=1}^{2A^*} (x - x_a^*)^2. \end{aligned} \quad (3.23)$$

The factor x^4 is introduced for convenience as we shall see below. For $\tilde{F}_4(x), \hat{F}_4(x)$ there are in total $4\tilde{A} + 4\hat{A} + 2A^*$ degrees of freedom.

Asymptotics. At $x = \infty$ the curve behaves as $y(x) \sim x$ and at $x = 0$ as $y(x) \sim 1/x$. This is achieved by the following range of exponents in the polynomials

$$\begin{aligned} \tilde{F}_k(x) &= *x^{4\tilde{A}+4A^*+8-k} + \dots + *x^k, \\ \hat{F}_k(x) &= *x^{4\hat{A}+4A^*+8-k} + \dots + *x^k. \end{aligned} \quad (3.24)$$

We can now count the remaining number of free coefficients. In $\tilde{F}_k(x), \hat{F}_k(x)$, $k < 4$, there are $4\tilde{A} + 4A^* + 9 - 2k$ and $4\hat{A} + 4A^* + 9 - 2k$ degrees of freedom, respectively. This leaves $20\tilde{A} + 20\hat{A} + 34A^* + 48$ relevant coefficients in total.

Unimodularity. The unimodularity condition $\hat{y}_1 + \hat{y}_2 + \hat{y}_3 + \hat{y}_4 = \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3 + \tilde{y}_4$ is imposed as a relation of the two leading coefficients of the algebraic equation

$$\frac{\tilde{F}_3(x)}{\tilde{F}_4(x)} = \frac{\hat{F}_3(x)}{\hat{F}_4(x)}. \quad (3.25)$$

This requires

$$\begin{aligned} \tilde{F}_3(x) &= F_3^*(x) \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^+) \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^-), \\ \hat{F}_3(x) &= F_3^*(x) \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^+) \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^-), \end{aligned} \quad (3.26)$$

with some polynomial

$$F_3^*(x) = *x^{4A^*+5} + \dots + *x^3. \quad (3.27)$$

It reduces the number of degrees of freedom by $4\tilde{A}+4\hat{A}+4A^*+3$ to $16\tilde{A}+16\hat{A}+30A^*+45$.

Symmetry. The symmetry $y(1/x) = y(x)$ is realized by the conditions

$$\begin{aligned} \tilde{F}_k(1/x) &= x^{-4\tilde{A}-4A^*-8} \tilde{F}_k(x), \\ \hat{F}_k(1/x) &= x^{-4\hat{A}-4A^*-8} \hat{F}_k(x), \\ F_3^*(1/x) &= x^{-4A^*-8} F_3^*(1/x). \end{aligned} \quad (3.28)$$

This yields $8\tilde{A}+8\hat{A}+15A^*+19$ constraints and leaves $8\tilde{A}+8\hat{A}+15A^*+26$ coefficients.

Singularities. We have to group up the residues at $x = \pm 1$ according to Sec. 2.6: Out of the 16 residues, there should only be 4 independent ones. This gives 12 constraints, but two of them have already been imposed by the unimodularity condition. As the singularities are at the fixed points $x = \pm 1$ of the symmetry $x \mapsto 1/x$, all 10 constraints can be imposed independently. This leaves $8\tilde{A}+8\hat{A}+15A^*+16$ degrees of freedom.

Unphysical Branch Points. In addition to the physical branch points at $\tilde{x}_a^\pm, \hat{x}_a^\pm$ the algebraic curve might have further ones. Generically, these singularities are square roots in contrast to the physical one which are inverse square roots. We can remove them using a condition of the discriminants²¹

$$\begin{aligned} \tilde{R} = & -4\tilde{F}_1^2\tilde{F}_2^3\tilde{F}_4 + 16\tilde{F}_0\tilde{F}_2^4\tilde{F}_4 - 27\tilde{F}_1^4\tilde{F}_4^2 + 144\tilde{F}_0\tilde{F}_1^2\tilde{F}_2\tilde{F}_4^2 - 128\tilde{F}_0^2\tilde{F}_2^2\tilde{F}_4^2 + 256\tilde{F}_0^3\tilde{F}_4^3 \\ & + 18\tilde{F}_1^3\tilde{F}_2\tilde{F}_3\tilde{F}_4 - 80\tilde{F}_0\tilde{F}_1\tilde{F}_2^2\tilde{F}_3\tilde{F}_4 - 192\tilde{F}_0^2\tilde{F}_1\tilde{F}_3\tilde{F}_4^2 - 6\tilde{F}_0\tilde{F}_1^2\tilde{F}_3^2\tilde{F}_4 + 144\tilde{F}_0^2\tilde{F}_2\tilde{F}_3^2\tilde{F}_4 \\ & + \tilde{F}_1^2\tilde{F}_2^2\tilde{F}_3^2 - 4\tilde{F}_0\tilde{F}_2^3\tilde{F}_3^2 - 4\tilde{F}_1^3\tilde{F}_3^3 + 18\tilde{F}_0\tilde{F}_1\tilde{F}_2\tilde{F}_3^3 - 27\tilde{F}_0^2\tilde{F}_3^4 \end{aligned} \quad (3.29)$$

and similarly for \hat{R} . The discriminants measure the product of squared distances of solutions $\tilde{y}_k(x)$ or $\hat{y}_k(x)$. A single root of $\tilde{R}(x) = 0$ or $\hat{R}(x) = 0$ thus implies a square root behavior which can only occur at $x = \tilde{x}_a^\pm$ or $x = \hat{x}_a^\pm$. The discriminants must therefore have the form

$$\begin{aligned} \tilde{R}(x) &= x^{12}(x^2 - 1)^4 \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^+) \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^-) \tilde{Q}(x)^2, \\ \hat{R}(x) &= x^{12}(x^2 - 1)^4 \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^+) \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^-) \hat{Q}(x)^2. \end{aligned} \quad (3.30)$$

It is clear that \tilde{x}_a^\pm and \hat{x}_a^\pm are roots, because all terms in (3.29) contain \tilde{F}_4 or \tilde{F}_3 . Noting the generic form of the discriminants

$$\begin{aligned} \hat{R}(x) &= *x^{24\hat{A}+24A^*+36} + \dots + *x^{12}, \\ \tilde{R}(x) &= *x^{24\tilde{A}+24A^*+36} + \dots + *x^{12}. \end{aligned} \quad (3.31)$$

²¹We could also use the equivalent condition: All solutions to $dF = 0$ are on the curve unless there is a physical singularity at this value of x . However, it is not quite clear how to count the number of constraints from this condition.

together with the inversion symmetry we find $5\tilde{A} + 5\hat{A} + 12A^* + 8$ constraints and $3\tilde{A} + 3\hat{A} + 3A^* + 8$ remaining degrees of freedom.

Single Poles and A-Cycles. We need to remove all the single poles and A-cycles from the curve $y(x)$ which would otherwise give rise to undesired logarithmic behavior in the quasi-momentum when restoring the quasi-momentum from its derivative. The symmetry $x \mapsto 1/x$ allows for 8 independent single poles in $y(x)$ at $x = \pm 1$. There are $\tilde{A} + \hat{A}$ independent A-cycles around bosonic cuts. Fermionic singularities contribute $2A^*$ independent single poles: one for \tilde{y} and one for \hat{y} at each $x = x_a^*$ modulo inversion symmetry. Among all these single poles and A-cycles, there are 4 relations from the sum over all residues, one for each pair of sheets related by the symmetry. In total this yields $\tilde{A} + \hat{A} + 2A^* + 4$ constraints and leaves $2\tilde{A} + 2\hat{A} + A^* + 4$ coefficients.

B-Periods. For each bosonic cut and for each fermionic singularity there is a B-period which must be integral. Furthermore, for each pair of sheets related by the symmetry, the B-period connecting 0 and ∞ must also be integral. Due to the unimodularity condition, only three of these periods are independent. In total we obtain $\tilde{A} + \hat{A} + A^* + 3$ constraints and are left with $\tilde{A} + \hat{A} + 1$ degrees of freedom.

Hypercharge. One degree of freedom corresponds to an irrelevant shift of the Lagrange multiplier, c.f. Sec. 2.4. The final number of moduli for admissible curves is $\tilde{A} + \hat{A}$.

3.3 Mode Numbers and Fillings

We will now associate each of the $\tilde{A} + \hat{A}$ moduli of the curve to one parameter per pair of bosonic cuts. We define the *filling* of an S^5 -cut $\tilde{\mathcal{C}}_a$ connecting sheets \tilde{k}_a and \tilde{l}_a as

$$\tilde{K}_a = -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\tilde{\mathcal{C}}_a} dx \left(1 - \frac{1}{x^2}\right) \tilde{p}_{\tilde{k}_a}(x) = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\tilde{\mathcal{C}}_a} \left(x + \frac{1}{x}\right) d\tilde{p}_{\tilde{k}_a}. \quad (3.32)$$

Our definition uses the sheet \tilde{k}_a , alternatively we might use \tilde{l}_a and invert the sign. Equivalently, we define the filling for an AdS_5 -cut $\hat{\mathcal{C}}_a$, but now using the sheet \hat{l}_a

$$\hat{K}_a = -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\hat{\mathcal{C}}_a} dx \left(1 - \frac{1}{x^2}\right) \hat{p}_{\hat{l}_a}(x) = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\hat{\mathcal{C}}_a} \left(x + \frac{1}{x}\right) d\hat{p}_{\hat{l}_a}. \quad (3.33)$$

The corresponding definition using the sheet \hat{k}_a would require an opposite sign. For completeness, we also define a filling for fermionic singularities x_a^*

$$K_a^* = -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a^*} dx \left(1 - \frac{1}{x^2}\right) \tilde{p}_{k_a^*}(x) = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a^*} \left(x + \frac{1}{x}\right) d\tilde{p}_{k_a^*}. \quad (3.34)$$

which we could also write using $\hat{p}_{l_a^*}$. It is not an independent modulus and it measures the residue at x_a^* .

In addition to the fillings, a curve is specified by the mode numbers

$$\tilde{n}_a = \frac{1}{2\pi} \int_{\tilde{\mathcal{B}}_a} dp, \quad \hat{n}_a = \frac{1}{2\pi} \int_{\hat{\mathcal{B}}_a} dp, \quad n_a^* = \frac{1}{2\pi} \int_{\mathcal{B}_a^*} dp. \quad (3.35)$$

These are discrete parameters and therefore not count as moduli. Note that the B-periods all start at $x = \infty$ on sheet $\tilde{k}_a, \hat{k}_a, k_a^*$ and end at $x = \infty$ on sheet $\tilde{l}_a, \hat{l}_a, l_a^*$, respectively. Furthermore, there is one overall winding number defined as

$$m = \frac{1}{2\pi} \int_{\infty}^0 \tilde{p}_{1,2} = -\frac{1}{2\pi} \int_{\infty}^0 \tilde{p}_{3,4}. \quad (3.36)$$

It is defined through the S^5 -part of the curve and there is no corresponding quantity for the AdS_5 -part, because there cannot be windings in the non-compact time direction of the universal covering of AdS_5 [28].

In most cases, the fillings give the right number of moduli, but for $m = 0$ there is a constraint among the fillings as we shall see below. Therefore, let us introduce one further modulus which we call the *length*²²

$$L = \frac{\sqrt{\lambda}}{16\pi^2 i} \oint_{+1} dx \sum_{k=1}^4 \varepsilon_k \tilde{p}_k + \frac{\sqrt{\lambda}}{16\pi^2 i} \oint_{-1} dx \sum_{k=1}^4 \varepsilon_k \tilde{p}_k + \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} \frac{dx}{x^2} \sum_{k=1}^4 \varepsilon_k \tilde{p}_k. \quad (3.37)$$

Note that we use only half of the $2A$ cuts for the definition of length, one from each pair related by inversion symmetry. This definition depends on which of the two cuts we select from each pair and is therefore ambiguous. In a particular limit, however, this choice is obvious as we shall see in Sec. 3.7. The length is related to the fillings by the constraint²³

$$mL = \sum_{a=1}^A n_a K_a \quad (3.38)$$

which means that among $\{L, K_a\}$ there are only $\tilde{A} + \hat{A}$ independent continuous parameters: $\tilde{A} + \hat{A} - 1$ independent fillings K_a and the length L . To derive it, consider the integral

$$\begin{aligned} 0 &= \frac{\sqrt{\lambda}}{32\pi^3 i} \oint_{\infty} dx \sum_{k=1}^4 (\tilde{p}_k^2(x) - \hat{p}_k^2(x)) \\ &= \frac{\sqrt{\lambda}}{32\pi^3 i} \oint_{+1} dx \sum_{k=1}^4 (\tilde{p}_k^2(x) - \hat{p}_k^2(x)) + \frac{\sqrt{\lambda}}{32\pi^3 i} \oint_{-1} dx \sum_{k=1}^4 (\tilde{p}_k^2(x) - \hat{p}_k^2(x)) \\ &\quad + \sum_{a=1}^{2A} \frac{\sqrt{\lambda}}{32\pi^3 i} \oint_{\mathcal{A}_a} dx \sum_{k=1}^4 (\tilde{p}_k^2(x) - \hat{p}_k^2(x)) \\ &= mL - \sum_{a=1}^A n_a K_a. \end{aligned} \quad (3.39)$$

²²The term ‘length’ is due to analogy with spin chains. For an alternative approach to identifying this conserved charge in the sigma model, see [43].

²³This constraint reveals the ambiguity of L : For some cuts the mode numbers and fillings of the mirror cut are related by $n_{A+a} = 2m - n_a$, $K_{A+a} = -K_a$. If we interchange the cut \mathcal{C}_a with its mirror image \mathcal{C}_{A+a} , L changes by $2K_a$.

The first integral is zero due to $p(x) \sim 1/x$ at $x = \infty$. We then split up the contour of integration around the singularities and cuts. To obtain the last line, we split up the integrals around $x = \pm 1$ evenly in two and also split up the sum $\sum_{a=1}^{2A}$ into $\sum_{a=1}^A$ and $\sum_{a=A+1}^{2A}$. Then we transform half of the integrals to $1/x$

$$\int_{\mathcal{A}_{A+a}} dx f(x) = - \int_{\mathcal{A}_a} \frac{dx}{x^2} f(1/x) \quad (3.40)$$

and use the inversion symmetry

$$\sum_{k=1}^4 (\tilde{p}_k^2(1/x) - \hat{p}_k^2(1/x)) = \sum_{k=1}^4 (\tilde{p}_k^2(x) - \hat{p}_k^2(x)) - 4\pi m \sum_{k=1}^4 \varepsilon_k \tilde{p}_k(x) + 16\pi^2 m^2 \quad (3.41)$$

to transform them back. The terms proportional to m^2 drop out from the integrals, they contain no residue, while the terms multiplying m sum up to L . The remaining integrals around $x = \pm 1$

$$\frac{\sqrt{\lambda}}{64\pi^3 i} \oint_{\pm 1} dx \left(1 - \frac{1}{x^2}\right) \sum_{k=1}^4 (\tilde{p}_k^2(x) - \hat{p}_k^2(x)) = 0. \quad (3.42)$$

sum up to zero as discussed in Sec. 2.7. In the final step we have employed the identity

$$\frac{\sqrt{\lambda}}{32\pi^3 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \sum_{k=1}^4 (\tilde{p}_k^2(x) - \hat{p}_k^2(x)) = -n_a K_a \quad (3.43)$$

which one obtains after pulling the contour \mathcal{A}_a tightly around the cut \mathcal{C}_a . Then the integrand $p^2(x + \epsilon) - p^2(x - \epsilon)$ can be split into symmetric and antisymmetric parts. The antisymmetric part is equal on two sheets up to a sign. The symmetric parts then combine using (3.12,3.13) and yield $2\pi n_a$. The remaining integral is the filling.

A more direct way to derive the constraint uses the Riemann bilinear identity

$$\frac{1}{2\pi i} \sum_a \left(\oint_{\mathcal{A}_a} dp \int_{\mathcal{B}_a} dq - \int_{\mathcal{B}_a} dp \oint_{\mathcal{A}_a} dq \right) = \frac{1}{2\pi i} \sum_a \text{Res}_a(p dq) \quad (3.44)$$

valid for any curve with a set of independent cycles $\mathcal{A}_a, \mathcal{B}_a$ and two arbitrary holomorphic differentials dp, dq . Let us briefly sketch the proof: We take as p the quasi-momentum and $dq = p dx$ and count the S^5 -part and AdS_5 -parts with opposite signs. The first product of integrals will be zero due to (3.8). According to (3.11,3.32,3.33) the second product of integrals leads to the sum $\sum_a n_a K_a$ over the bosonic cuts when the symmetry is taken into account as explained above. The sum of residues of p^2 yields the contributions from the fermions using (3.14,3.34). The residues from $x = \pm 1$ cancel and the term mL appears during symmetrization as above.

3.4 Moduli of String Solutions

At this point we briefly summarize our results on the number of moduli and compare it to the general solution of strings in flat space or on plane waves. We have found

one continuous modulus, the filling K_a , and one discrete parameter, n_a , per pair of cuts (related by inversion symmetry). Furthermore we need to specify which of the 4|4 sheets are connected by the cut through k_a, l_a . The situation for fermionic poles is similar, only that their filling is not an independent parameter. In addition, there is one continuous global modulus, the length L , and one discrete global parameter, m , but also one global constraint which relates K_a, n_a, L, m . Note that we have discarded λ which can be considered as an external parameter.

The classification for (classical) strings in flat space or on plane waves is similar: Consider a solution with only a finite number of active string modes. Let us furthermore assume a light-cone gauge to focus on the physical excitations. Then each mode is described by its mode number (n_a), amplitude (K_a) and orientation (k_a, l_a) where we have indicated in brackets the corresponding quantities in our sigma model. The amplitudes of fermions cannot be specified by regular numbers and thus should not be counted as continuous moduli. One overall level matching constraint relates the amplitudes and mode numbers (K_a, n_a). The string tension (λ) will again be considered external. The only difference between strings in flat space and our model is the lack of a modulus describing the effective curvature (L) and a parameter describing winding (m).

While the relation between amplitudes and fillings as well as integers n and mode numbers is obvious, the relation between sheets and orientation of the string needs further explanations. For cuts related to S^5 we see that there are 6 pairs of sheets and thus 6 choices for $(\tilde{k}_a, \tilde{l}_a)$. Similarly for AdS^5 . Fermions have to connect one sheet of each type and thus there are 16 choices. It thus seems that there are $(6+6)|16$ orientations. There is however a further criterion which we use to distinguish cuts and poles. We denote the cuts/poles with $\varepsilon_k \neq \varepsilon_l$ as *physical*. The cuts/poles with $\varepsilon_k = \varepsilon_l$ are considered *auxiliary*. The explanation for this classification is that precisely the physical cuts/poles appear within the combination $q(x)$ in (2.73) which is used to define the local charges (and also the energy shift, c.f. the following subsection). Among the 6 types of bosonic cuts each, there are 4 physical and 2 auxiliary ones. The 16 types of fermionic poles split up evenly into 8 physical and 8 auxiliary ones. Thus the counting of orientations for physical modes, $(4+4)|8$, is as expected for a superstring.

In conclusion we see that the moduli of admissible curves are in one to one correspondence to the moduli describing closed superstrings in flat space. We expect that the number of moduli and their types should be mostly independent of the background. The only relevant properties for the enumeration of moduli (open/closed, bosonic/supersymmetric, number of spacetime dimensions, smoothness of the target space, ...) are the same in both theories. We take this as compelling evidence that all admissible curves, as discussed in this section, indeed correspond to at least one string solution. We thus believe that we have not missed a relevant characteristic feature in Sec. 2 for the construction of admissible curves and that our classification is complete.²⁴

²⁴We only refer to the action variables of string solutions. Of course, the (time-dependent) angle variables are not described by the algebraic curve. According to standard lore, they correspond to a set of marked point on the Jacobian of the curve.

3.5 Global Charges

Here we shall relate the global charges of $\text{PSU}(2, 2|4)$ to the fillings. Let us concentrate on S^5 at first and define global fillings

$$\begin{aligned}\tilde{K}_1 &= - \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{3}{4}\tilde{p}_1 - \frac{1}{4}\tilde{p}_2 - \frac{1}{4}\tilde{p}_3 - \frac{1}{4}\tilde{p}_4\right), \\ \tilde{K}_2 &= - \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{1}{2}\tilde{p}_1 + \frac{1}{2}\tilde{p}_2 - \frac{1}{2}\tilde{p}_3 - \frac{1}{2}\tilde{p}_4\right), \\ \tilde{K}_3 &= - \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{1}{4}\tilde{p}_1 + \frac{1}{4}\tilde{p}_2 + \frac{1}{4}\tilde{p}_3 - \frac{3}{4}\tilde{p}_4\right).\end{aligned}\tag{3.45}$$

These can also be represented as a sum of fillings \tilde{K}_a of the individual cuts and residues K_a^* of fermionic poles. We will not do this explicitly, as there are too many pairs of sheets and thus too many types of cuts. The Dynkin labels $[\tilde{r}_1, \tilde{r}_2, \tilde{r}_3]$ of $\text{SU}(4)$ are given by the following combinations

$$\tilde{r}_j = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\infty} dx (\tilde{p}_j(x) - \tilde{p}_{j+1}(x)).\tag{3.46}$$

Their relation to the global fillings is as follows

$$\begin{aligned}\tilde{r}_1 &= \tilde{K}_2 - 2\tilde{K}_1, & \tilde{K}_1 &= \frac{1}{2}L - \frac{3}{4}\tilde{r}_1 - \frac{1}{2}\tilde{r}_2 - \frac{1}{4}\tilde{r}_3, \\ \tilde{r}_2 &= L - 2\tilde{K}_2 + \tilde{K}_1 + \tilde{K}_3, & \tilde{K}_2 &= L - \frac{1}{2}\tilde{r}_1 - \tilde{r}_2 - \frac{1}{2}\tilde{r}_3, \\ \tilde{r}_3 &= \tilde{K}_2 - 2\tilde{K}_3, & \tilde{K}_3 &= \frac{1}{2}L - \frac{1}{4}\tilde{r}_1 - \frac{1}{2}\tilde{r}_2 - \frac{3}{4}\tilde{r}_3.\end{aligned}\tag{3.47}$$

To derive these, it is convenient to make use of the inversion symmetry, c.f. the previous subsection.

For AdS_5 the results are very similar. Again we define the global fillings

$$\begin{aligned}\hat{K}_1 &= \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{3}{4}\hat{p}_1 - \frac{1}{4}\hat{p}_2 - \frac{1}{4}\hat{p}_3 - \frac{1}{4}\hat{p}_4\right), \\ \hat{K}_2 &= \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{1}{2}\hat{p}_1 + \frac{1}{2}\hat{p}_2 - \frac{1}{2}\hat{p}_3 - \frac{1}{2}\hat{p}_4\right), \\ \hat{K}_3 &= \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{1}{4}\hat{p}_1 + \frac{1}{4}\hat{p}_2 + \frac{1}{4}\hat{p}_3 - \frac{3}{4}\hat{p}_4\right),\end{aligned}\tag{3.48}$$

which we might write as sums of the individual fillings. Then the Dynkin labels are given by

$$\hat{r}_j = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\infty} dx (\hat{p}_{j+1}(x) - \hat{p}_j(x)).\tag{3.49}$$

and related to the global fillings by

$$\begin{aligned}\hat{r}_1 &= \hat{K}_2 - 2\hat{K}_1, & \hat{K}_1 &= -\frac{1}{2}L - \frac{1}{2}\delta E - \frac{3}{4}\hat{r}_1 - \frac{1}{2}\hat{r}_2 - \frac{1}{4}\hat{r}_3, \\ \hat{r}_2 &= -L - \delta E - 2\hat{K}_2 + \hat{K}_1 + \hat{K}_3, & \hat{K}_2 &= -L - \delta E - \frac{1}{2}\hat{r}_1 - \hat{r}_2 - \frac{1}{2}\hat{r}_3, \\ \hat{r}_3 &= \hat{K}_2 - 2\hat{K}_3, & \hat{K}_3 &= -\frac{1}{2}L - \frac{1}{2}\delta E - \frac{1}{4}\hat{r}_1 - \frac{1}{2}\hat{r}_2 - \frac{3}{4}\hat{r}_3.\end{aligned}\tag{3.50}$$

Here we have introduced a new quantity δE , the *energy shift*

$$\delta E = \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} \frac{dx}{x^2} \sum_{k=1}^4 (-\varepsilon_k \tilde{p}_k + \varepsilon_k \hat{p}_k) = - \sum_{a=1}^A \frac{\sqrt{\lambda}}{4\pi^2 i} \oint_{\mathcal{A}_a} \frac{dx}{x^2} q(x)\tag{3.51}$$

with $q(x)$ defined in (2.73). When we write \hat{r}_2 in terms of the AdS_5 energy E

$$E = -r_2 - \frac{1}{2}r_1 - \frac{1}{2}r_3 = L + \hat{K}_2 + \delta E\tag{3.52}$$

we see that δE is indeed the energy shift when $L + \hat{K}_2$ is interpreted as the bare energy. Finally, we introduce the global fermionic filling

$$K^* = - \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \sum_{k=1}^4 \left(\frac{1}{2}\tilde{p}_k + \frac{1}{2}\hat{p}_k\right).\tag{3.53}$$

It is related to the hypercharge eigenvalue r^*

$$r^* = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\infty} dx \sum_{k=1}^4 \left(\frac{1}{2}\tilde{p}_k(x) + \frac{1}{2}\hat{p}_k(x)\right) = 2B - K^*.\tag{3.54}$$

We have introduced a charge B which is related to the Lagrange multiplier, see Sec. 2.4. Under the symmetry it transforms as $B \mapsto B + \frac{1}{2}\nu\sqrt{\lambda}$.

3.6 Superstrings on $AdS_3 \times S^3$

Let us consider solutions of the supersymmetric $AdS_5 \times S^5$ sigma model which extend only over a supersymmetric $AdS_3 \times S^3$ subspace, which in fact is given by the group manifold $PSU(1,1|2)$. For this class of solutions, the algebraic curve will split into two disconnected parts. The first component consists of \tilde{p}_2, \tilde{p}_3 and \hat{p}_1, \hat{p}_4 and the other component consists of the remaining four sheets. There are no branch cuts or fermionic poles connecting the two parts. Both components are isomorphic to algebraic curves obtained from the $PSU(1,1|2)$ sigma model [44]. One of them corresponds to the monodromy in the fundamental representation, the other one to the monodromy in the antifundamental representation. These two curves are not unrelated, for a sigma model on a group manifold they should map into each other under inversion $x \mapsto 1/x$. Indeed, this is precisely what the $AdS_5 \times S^5$ sigma model implies, see Sec. 2.5. There are several conceptual differences which make it interesting to consider the $AdS_3 \times S^3$ model separately.

First of all, the $AdS_3 \times S^3$ model leads to one algebraic curve without inversion symmetry (or, equivalently, two related algebraic curves) whereas the full $AdS_5 \times S^5$

model has only one self-symmetric curve. This also means that we can distinguish between a cut and its image under inversion: They reside on different components of the algebraic curve and we shall consider only one component. The definitions of length (3.37) and energy shift (3.51) thus become natural and unambiguous. In fact they become two of the global charges. Together with the spin S on AdS_3 and spin J_2 on S^3 they are the four charge eigenvalues of the isometry group $PSU(1,1|2) \times PSU(1,1|2)$. Again we can express the global charges through the fillings

$$\begin{aligned} J_2 = \tilde{K} &= - \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{1}{2}\tilde{p}_2 - \frac{1}{2}\tilde{p}_3\right), \\ S = \hat{K} &= \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{A}_a} dx \left(1 - \frac{1}{x^2}\right) \left(\frac{1}{2}\hat{p}_1 - \frac{1}{2}\hat{p}_4\right), \end{aligned} \quad (3.55)$$

Here we have defined J_2 and S to match the fillings. The expansion of the quasi-momentum at $x = \infty$ is related to the charges of one of the global $PSU(1,1|2)$ factors

$$\begin{aligned} \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\infty} dx (\tilde{p}_2 - \tilde{p}_3) &= L - 2\tilde{K} = J_1 - J_2, \\ \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\infty} dx (\hat{p}_4 - \hat{p}_1) &= -L - \delta E - 2\hat{K} = -E - S. \end{aligned} \quad (3.56)$$

For convenience, we have defined the spin J_1 and energy E to replace the length and energy shift as follow

$$L = J_1 + J_2, \quad \delta E = E - L - S. \quad (3.57)$$

The expansion at $x = 0$ relates to the charges of the other global $PSU(1,1|2)$ factor

$$\begin{aligned} \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_0 \frac{dx}{x^2} (\tilde{p}_2 - \tilde{p}_3) &= L = J_1 + J_2, \\ \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_0 \frac{dx}{x^2} (\hat{p}_4 - \hat{p}_1) &= -L - \delta E = -E + S. \end{aligned} \quad (3.58)$$

These expressions agree precisely with the rank-one subsectors of $\mathbb{R} \times S^3$ and $AdS_3 \times S^1$ considered in [17, 28].

3.7 The Frolov-Tseytlin Limit

In this section we shall discuss the Frolov-Tseytlin limit $L/\sqrt{\lambda} \rightarrow \infty$ of the curve.²⁵ In this limit, half of the cuts and poles approach $x = \infty$ and half of them approach $x = 0$.²⁶ Let us label those cuts and poles which escape to $x = \infty$ by $a = 1, \dots, A$, those which approach $x = 0$ will be labelled by $a = A + 1, \dots, 2A$. Therefore, there is a natural choice for those cuts which contribute to the definition of the length in (3.37).

²⁵At the level of the action and the Hamiltonian, this limit was studied in [27].

²⁶Solutions with self-symmetric cuts do not have a proper Frolov-Tseytlin limit.

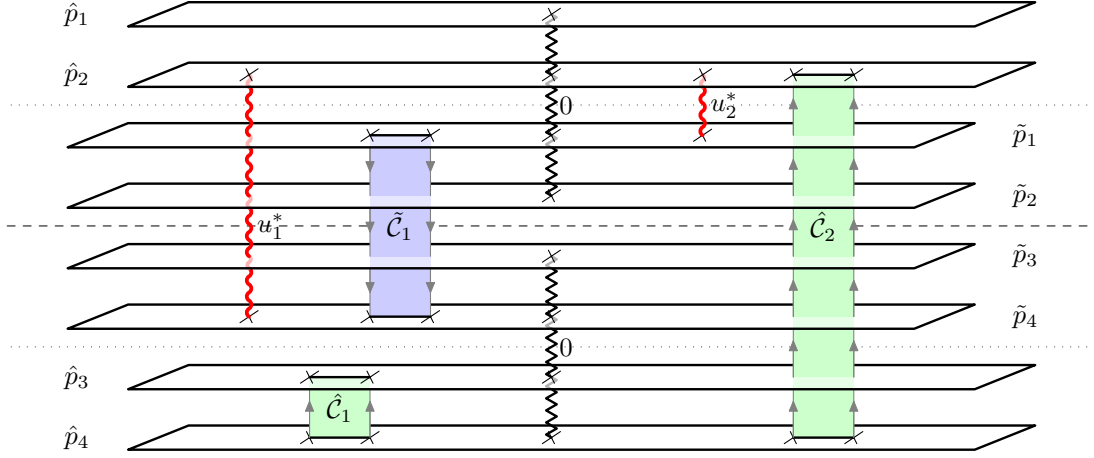


Figure 5: The Frolov-Tseytlin limit of the configuration of cuts and poles in Fig. 3. All inverse cuts and poles as well as the poles at $x = \pm 1$ have been scaled to $u = 0$ and absorbed into an effective pole.

We will define for convenience a rescaled variable²⁷

$$u = \frac{\sqrt{\lambda}}{4\pi} x. \quad (3.59)$$

In other words, in the u -plane all cuts and poles with $a = 1, \dots, A$ approach finite values of u while their images approach $u = 0$. Similarly, the poles at $x = \pm 1$ approach $u = 0$, see Fig. 5. This means that the point $u = 0$ is special and $p(u)$ near $u = 0$ is not directly related to $p(x)$ near $x = 0$, but there are contributions from the cuts and poles. To understand the expansion of $p(u)$ at $u = 0$ we shall define a contour \mathcal{C} in the x -plane which encircles the poles at $x = \pm 1$ and all the cuts and poles with $a = A + 1, \dots, 2A$. Equivalently, this may be considered a contour which excludes $x = \infty$ and all the cuts and poles with $a = 1, \dots, A$. After rescaling \mathcal{C} merely encircles the point $u = 0$ in the u -plane which can be used to obtain the expansion of $p(u)$ according to the formula²⁸

$$\frac{\partial^{r-1} p_k}{\partial u^{r-1}}(0) = \left(\frac{4\pi L}{\sqrt{\lambda}} \right)^{r-1} \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dx}{x^r} p_k(x). \quad (3.60)$$

Using the identities and definitions in Sec. 3.3,3.5 we find the singular behavior of all sheets at $u = 0$

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\mathcal{C}} \tilde{p}_k(x) dx &= \frac{2\pi}{\sqrt{\lambda}} (B + \varepsilon_k L) - \sum_{a=1}^A \frac{1}{2\pi i} \oint_{\mathcal{A}_a} \frac{dx}{x^2} \tilde{p}_k(x), \\ \frac{1}{2\pi i} \oint_{\mathcal{C}} \hat{p}_k(x) dx &= \frac{2\pi}{\sqrt{\lambda}} (B + \varepsilon_k L + \varepsilon_k \delta E) - \sum_{a=1}^A \frac{1}{2\pi i} \oint_{\mathcal{A}_a} \frac{dx}{x^2} \hat{p}_k(x) \end{aligned} \quad (3.61)$$

²⁷This relationship needs to be refined at higher orders in $L/\sqrt{\lambda}$ [17, 25].

²⁸This formula explains why there are two different transfer matrices $T(x), \bar{T}(u)$ in [25]. The transfer matrix $\bar{T}(u)$ is the suitable one for finite g , while $T(x)$ is the effective one according to this formula. It should also be useful to understand the relationship between conserved local charges in string theory and gauge theory in [45].

and the first few moments of the generator of local charges (2.73)

$$\begin{aligned}\frac{1}{2\pi i} \oint_{\mathcal{C}} dx q(x) &= -\frac{2\pi \delta E}{\sqrt{\lambda}}, \\ \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dx}{x} q(x) &= 2\pi m, \\ \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dx}{x^2} q(x) &= \frac{2\pi \delta E}{\sqrt{\lambda}}.\end{aligned}\tag{3.62}$$

Now we note that the energy shift (3.51)

$$\delta E = -\sum_{a=1}^A \frac{\sqrt{\lambda}}{4\pi^2 i} \oint_{\mathcal{A}_a} \frac{dx}{x^2} q(x) = -\frac{\lambda}{8\pi^2 L} \sum_{a=1}^A \frac{1}{2\pi i} \oint_{\mathcal{A}_a} \frac{du}{u^2} q(u).\tag{3.63}$$

is of order $\mathcal{O}(\lambda/L)$. After rescaling we obtain the singular behavior of $p(u)$ at $u = 0$ from (3.60,3.61)

$$\tilde{p}_k(u) \sim \hat{p}_k(u) \sim \frac{1}{u} \left(\frac{\varepsilon_k}{2} + \frac{B}{2L} \right)\tag{3.64}$$

and the first few local charges from (3.60,3.62)

$$q(u) = 2\pi m + \frac{8\pi^2 L}{\lambda} \delta E u + \mathcal{O}(u^2).\tag{3.65}$$

In particular, the momentum constraint is $q(0) = 2\pi\mathbb{Z}$ while the individual sheets $p_k(u)$ no longer have a fixed finite value at $u = 0$.

The above curve apparently is the spectral curve of the supersymmetric Landau-Lifshitz model in [32].²⁹ This model is related to the coherent state approach to gauge theory [27]. Unlike the curve of the full superstring, this curve has only one singular point at $u = 0$ and thus seems to be similar to the one of the classical Heisenberg magnet discussed in [17]. We expect the expansion of the function $q(u)$ around $u = 0$ to yield the local charges of the model, while the point $u = \infty$ should be related to Noether and multi-local charges.

4 Integral Representation of the Sigma Model

We can reformulate the algebraic curve in terms of a Riemann-Hilbert problem, i.e. as integral equations on some density functions. This formulation is similar to the thermodynamic limit of the Bethe equations for the gauge theory counterpart. It is thus suited well for a comparison of both theories, especially at higher loops (once the gauge theory results become available). We start by representing the various discontinuities of cuts of the algebraic curve by integrals over densities. We then match this representation to the properties derived earlier and thus fix several of the parameters. The remaining properties lead to equations which are of the same nature as the Bethe equations in the thermodynamic limit. In order to be more explicit, we specify the equations for a number of subsectors, while full equations are written out only in App. D. Finally, we compare to one-loop gauge theory and find complete agreement.

²⁹We thank A. Mikhailov for discussions on this point.

4.1 Parametrization of the Quasi-Momentum

The quasi-momentum $p(x)$ is a function with two sets of four sheets $\tilde{p}_k(x)$ and $\hat{p}_k(x)$, $k = 1, 2, 3, 4$. These are analytic functions of x except at the singular points $x = \pm 1$, a set of branch cuts and additional fermionic poles.

Ansatz. Let us construct a generic ansatz for $p(x)$: It is straightforward to incorporate the poles at $x = \pm 1$ with undetermined residues \tilde{a}_k^\pm and \hat{a}_k^\pm . The branch cuts and fermionic poles will be contained in several resolvents $G(x)$. All the resolvents $G(x)$ will be defined to vanish at $x = \infty$, consequently we should add an undetermined constant \tilde{b}_k, \hat{b}_k for each sheet.

The branch cuts connect two sheets of either $\tilde{p}(x)$ or $\hat{p}(x)$. The discontinuity of the branch cuts between sheets k and l is contained in the resolvent $\tilde{G}_{kl}(x)$ or $\hat{G}_{kl}(x)$. As the graded sum of all sheets p_k should be zero, the sum of discontinuities must cancel. In other words, \tilde{G}_{kl} and \hat{G}_{kl} must be antisymmetric in k, l . A fermionic pole appears on a sheet \tilde{p}_k and a sheet \hat{p}_l with the same residue for both sheets. They are contained in the resolvent $G_{kl}^*(x)$. We shall include only the cuts/poles with $a = 1, \dots, A$ in the resolvent $G(x)$. Their images with $a = A + 1, \dots, 2A$ under the inversion symmetry (3.3) will be incorporated by $G(1/x)$. This leaves some discrete arbitrariness of which cuts/poles belong to $G(x)$ and which to $G(1/x)$. Note that although the algebraic curve is invariant under such permutations, our interpretation of some quantities will have to change.

Taking the above constraints into account, we arrive at the following ansatz

$$\begin{aligned}\tilde{p}_k(x) &= \sum_{l=1}^4 \left(\tilde{G}_{kl}(x) - \tilde{G}_{k'l}(1/x) + G_{kl}^*(x) - G_{k'l}^*(1/x) \right) + \frac{\tilde{a}_k^+}{x-1} + \frac{\tilde{a}_k^-}{x+1} + \tilde{b}_k, \\ \hat{p}_k(x) &= \sum_{l=1}^4 \left(\hat{G}_{lk}(x) - \hat{G}_{lk'}(1/x) + G_{lk}^*(x) - G_{lk'}^*(1/x) \right) + \frac{\hat{a}_k^+}{x-1} + \frac{\hat{a}_k^-}{x+1} + \hat{b}_k.\end{aligned}\quad (4.1)$$

where the permutation k' of sheets is defined in (2.58). We will now determine the constants using the known properties of $p(x)$.

Resolvents. The bosonic resolvents $\tilde{G}(x), \hat{G}(x)$ are defined in terms of the densities $\tilde{\rho}(x), \hat{\rho}(x)$ as follows

$$\tilde{G}_{kl}(x) = \int_{\tilde{C}_{kl}} \frac{dy \tilde{\rho}_{kl}(y)}{1 - 1/y^2} \frac{1}{y - x}, \quad \hat{G}_{kl}(x) = \int_{\hat{C}_{kl}} \frac{dy \hat{\rho}_{kl}(y)}{1 - 1/y^2} \frac{1}{y - x}. \quad (4.2)$$

The fermionic resolvent $G_{kl}^*(x)$ is given by a set of poles³⁰

$$G_{kl}^*(x) = \sum_{a=1}^{A_{kl}^*} \frac{\alpha_{kl,a}^*}{1 - 1/x_{kl,a}^{*2}} \frac{1}{x_{kl,a}^* - x}. \quad (4.3)$$

³⁰Strictly speaking the residues must be nilpotent numbers because they represent a product of two Grassmann odd numbers, but we can mostly ignore this fact.

All the resolvents vanish at $x = \infty$ and are analytic functions except on the curves \mathcal{C}_a or at the poles x_a^* . They are obviously single-valued, i.e. the cycles of dG around cuts/poles vanish. The filling of a cut (3.32,3.33,3.34) is given by

$$\tilde{K}_{kl,a} = \frac{\sqrt{\lambda}}{4\pi} \int_{\tilde{\mathcal{C}}_{kl,a}} dy \tilde{\rho}_{kl}(y), \quad \hat{K}_{kl,a} = \frac{\sqrt{\lambda}}{4\pi} \int_{\hat{\mathcal{C}}_{kl,a}} dy \hat{\rho}_{kl}(y), \quad K_{kl,a}^* = \frac{\sqrt{\lambda}}{4\pi} \alpha_{kl,a}^*. \quad (4.4)$$

Singularities. From Sec. 2.7 we know the general structure of the residues at $x = \pm 1$. The residues \tilde{a}_k^\pm for S^5 are linked to the residues \hat{a}_k^\pm for AdS_5 . Furthermore, all residues are paired. We can thus write them in terms of four independent constants $c_{1,2}, d_{1,2}$ using (2.60)

$$\tilde{a}_k^\pm = \hat{a}_k^\pm =: \frac{1}{2}c_1 \pm \frac{1}{2}c_2 + \frac{1}{2}d_1\varepsilon_k \pm \frac{1}{2}d_2\varepsilon_k. \quad (4.5)$$

Asymptotics. The asymptotics $p(x) \sim 1/x$ at $x = \infty$ fixes all the constants b

$$\tilde{b}_k = \sum_{l=1}^4 \left(\tilde{G}_{k'l}(0) + G_{k'l}^*(0) \right), \quad \hat{b}_k = \sum_{l=1}^4 \left(\hat{G}_{lk'}(0) + G_{lk'}^*(0) \right). \quad (4.6)$$

Unimodularity. The graded sum

$$\sum_{k=1}^4 (\tilde{p}_k(x) - \hat{p}_k(x)) = 0 \quad (4.7)$$

of all sheets indeed vanishes trivially by antisymmetry of $\tilde{G}_{kl}, \hat{G}_{kl}$ in k, l .

Symmetry. The inversion symmetry leads to the following expressions

$$\begin{aligned} \tilde{p}_k(1/x) &= -\tilde{p}_{k'}(x) - \tilde{a}_k^+ + \tilde{a}_k^- + \tilde{b}_k + \tilde{b}_{k'} \stackrel{!}{=} -\tilde{p}_{k'}(x) + 2\pi\varepsilon_k m, \\ \hat{p}_k(1/x) &= -\hat{p}_{k'}(x) - \hat{a}_k^+ + \hat{a}_k^- + \hat{b}_k + \hat{b}_{k'} \stackrel{!}{=} -\hat{p}_{k'}(x) \end{aligned} \quad (4.8)$$

with the permutation k' of sheets is defined in (2.58). When we substitute the above expressions for $\tilde{a}_k^\pm, \hat{a}_k^\pm$, we find

$$c_2 = \frac{1}{2} \sum_{k,l=1}^4 G_{kl}^*(0), \quad d_2 = \frac{1}{2} \sum_{k,l=1}^4 \varepsilon_k (\hat{G}_{lk}(0) + G_{lk}^*(0)) \quad (4.9)$$

as well as the (momentum) constraint

$$\frac{1}{2} \sum_{k,l=1}^4 \varepsilon_k (\tilde{G}_{kl}(0) + \hat{G}_{kl}(0) + G_{kl}^*(0) - G_{lk}^*(0)) = 2\pi m. \quad (4.10)$$

Length. We substitute \tilde{p}_k in the definition of length and obtain

$$L = \frac{\sqrt{\lambda}}{8\pi} \sum_{k=1}^4 \varepsilon_k (\tilde{a}_k^+ + \tilde{a}_k^-) - \frac{\sqrt{\lambda}}{4\pi} \sum_{k,l=1}^4 \varepsilon_k (\tilde{G}'_{kl}(0) + G_{kl}^{*'}(0)) \quad (4.11)$$

This leads to

$$d_1 = \frac{2\pi L}{\sqrt{\lambda}} + \frac{1}{2} \sum_{k,l=1}^4 \varepsilon_k (\tilde{G}'_{kl}(0) + G_{kl}^{*'}(0)). \quad (4.12)$$

Note that we have assumed that all cuts $a = 1, \dots, \tilde{A}$ are captured by $G(x)$ while the cuts $a = \tilde{A} + 1, \dots, 2\tilde{A}$ are captured by $G(1/x)$.

Hypercharge. The remaining constant c_1 corresponds to a shift of the Lagrange multiplier and is irrelevant. We shall express it by means of the hypercharge B defined in (3.54)

$$c_1 = \frac{1}{2} \sum_{k,l=1}^4 G_{kl}^{*'}(0) + \frac{2\pi B}{\sqrt{\lambda}}. \quad (4.13)$$

Energy Shift. We substitute p_k in the definition of energy shift and obtain

$$\delta E = \frac{\sqrt{\lambda}}{4\pi} \sum_{k,l=1}^4 \varepsilon_k (\tilde{G}'_{kl}(0) + G_{kl}^{*'}(0) - \hat{G}'_{lk}(0) - G_{lk}^{*'}(0)). \quad (4.14)$$

4.2 Integral Equations

Let us now assemble and simplify the various findings of the previous section. As a first step, we write $G(x)$, but not $G(1/x)$, $G(0)$ or $G'(0)$, in terms of the inversion-symmetric function

$$H_{kl}(x) := G_{kl}(x) + G_{kl}(1/x) - G_{kl}(0). \quad (4.15)$$

The terms in the integrand of H combine as follows

$$\frac{1}{1 - 1/y^2} \left(\frac{1}{y - x} + \frac{1}{y - 1/x} - \frac{1}{y} \right) = \frac{1}{(y + 1/y) - (x + 1/x)} \quad (4.16)$$

which means that we can write H as

$$H(x) := \int_{\mathcal{C}} \frac{du \rho(u)}{u - (x + 1/x)}, \quad (4.17)$$

where ρ transforms as a density, $dx \rho(x) = du \rho(u)$, under the map

$$u(x) = x + 1/x, \quad x(u) = \frac{1}{2}u + \frac{1}{2}u\sqrt{1 - 4/u^2}. \quad (4.18)$$

The conversion of $G(x)$ to $H(x)$ creates a few more instances of $G_{kl}(1/x)$ which turn out to pair up with existing instances of $G_{k'l}(1/x)$ in all cases. These can be rewritten by applying

$$f_k + f_{k'} = \frac{1}{2} \sum_{l=1}^4 f_l + \frac{1}{2} \varepsilon_k \sum_{l=1}^4 \varepsilon_l f_l \quad (4.19)$$

This identity for any f_k can easily be verified by evaluating it for all possible values $k = 1, 2, 3, 4$. It is convenient to introduce the following combinations of resolvents

$$\begin{aligned} \tilde{G}_{\text{sum}}(x) &= \frac{1}{2} \sum_{k,l=1}^4 \varepsilon_k (\tilde{G}_{kl}(x) + G_{kl}^*(x)), \\ \hat{G}_{\text{sum}}(x) &= \frac{1}{2} \sum_{k,l=1}^4 \varepsilon_k (\hat{G}_{lk}(x) + G_{lk}^*(x)), \\ G_{\text{sum}}^*(x) &= \frac{1}{2} \sum_{k,l=1}^4 G_{kl}^*(x), \\ G_{\text{mom}}(x) &= \tilde{G}_{\text{sum}}(x) - \hat{G}_{\text{sum}}(x) \end{aligned} \quad (4.20)$$

We can now write down the simplified quasi-momentum

$$\begin{aligned} \tilde{p}_k(x) &= \sum_{l=1}^4 (\tilde{H}_{kl}(x) + H_{kl}^*(x)) + \varepsilon_k \tilde{F}(x) + F^*(x), \\ \hat{p}_k(x) &= \sum_{l=1}^4 (\hat{H}_{lk}(x) + H_{lk}^*(x)) + \varepsilon_k \hat{F}(x) + F^*(x). \end{aligned} \quad (4.21)$$

All the terms which do not follow the regular pattern of resolvents H could be absorbed into three potentials F

$$\begin{aligned} \tilde{F}(x) &= \left(\frac{2\pi L}{\sqrt{\lambda}} + \tilde{G}'_{\text{sum}}(0) \right) \frac{1/x}{1 - 1/x^2} + \frac{\tilde{G}_{\text{sum}}(0)}{1 - 1/x^2} - \tilde{G}_{\text{sum}}(1/x) + G_{\text{mom}}(0), \\ \hat{F}(x) &= \left(\frac{2\pi L}{\sqrt{\lambda}} + \tilde{G}'_{\text{sum}}(0) \right) \frac{1/x}{1 - 1/x^2} + \frac{\hat{G}_{\text{sum}}(0)}{1 - 1/x^2} - \hat{G}_{\text{sum}}(1/x), \\ F^*(x) &= \left(\frac{2\pi B}{\sqrt{\lambda}} + G_{\text{sum}}^{*'}(0) \right) \frac{1/x}{1 - 1/x^2} + \frac{G_{\text{sum}}^*(0)}{1 - 1/x^2} - G_{\text{sum}}^*(1/x). \end{aligned} \quad (4.22)$$

It might be useful to note the transformation of the potentials F under the symmetry³¹

$$\begin{aligned} \tilde{F}(1/x) &= -\tilde{F}(x) - \tilde{H}_{\text{sum}}(x) + G_{\text{mom}}(0), \\ \hat{F}(1/x) &= -\hat{F}(x) - \hat{H}_{\text{sum}}(x), \\ F^*(1/x) &= -F^*(x) - H_{\text{sum}}^*(x). \end{aligned} \quad (4.23)$$

³¹The summed resolvents H_{sum} are defined in analogy to (4.20).

The integral equations (3.12,3.13) enforcing integrality of the B-periods (3.11,3.14) read

$$\begin{aligned}\tilde{p}_l(x) - \tilde{p}_k(x) &= 2\pi\tilde{n}_{kl,a} & \text{for } x \in \tilde{\mathcal{C}}_{kl,a}, \\ \hat{p}_l(x) - \hat{p}_k(x) &= 2\pi\hat{n}_{kl,a} & \text{for } x \in \hat{\mathcal{C}}_{kl,a}, \\ \check{p}_l(x) - \check{p}_k(x) &= 2\pi n_{kl,a}^* & \text{for } x = x_{kl,a}^*.\end{aligned}\tag{4.24}$$

These equations must be supplemented by the momentum constraint (4.10)

$$G_{\text{mom}}(0) = 2\pi m.\tag{4.25}$$

Note that the potential can appear in various combinations depending on the type of cut/pole. Let us denote those cuts/poles with $\varepsilon_k \neq \varepsilon_l$ as *physical*, the others are considered as *auxiliary*. The physical cuts are precisely the ones that contribute to G_{mom} which in turn contains the total momentum for the momentum constraint (4.10) and the energy shift (4.14). They connect sheets 1, 2 to sheet 3, 4 of either type. A physical cut is subject to the potential $2\tilde{F}$ or $2\hat{F}$ depending on whether it is of S^5 -type or AdS_5 -type. For an auxiliary bosonic cut there is no effective potential. For physical fermionic poles we get the potential $\tilde{F} + \hat{F}$ and $\tilde{F} - \hat{F}$ for auxiliary fermions.

The global charges are found at $x = \infty$, they are determined through the fillings K , the length L and the energy shift δE : The expansion of H gives the total filling K of the cuts in H

$$H(x) = -\frac{1}{x} \sum_{a=1}^A \frac{4\pi K_a}{\sqrt{\lambda}} + \mathcal{O}(1/x^2) = -\frac{1}{x} \frac{4\pi K}{\sqrt{\lambda}} + \mathcal{O}(1/x^2).\tag{4.26}$$

The expansion of F provides the length and the energy shift (4.14)

$$\tilde{F}(x) = \frac{1}{x} \frac{2\pi L}{\sqrt{\lambda}} + \mathcal{O}(1/x^2), \quad \hat{F}(x) = \frac{1}{x} \frac{2\pi (L + \delta E)}{\sqrt{\lambda}} + \mathcal{O}(1/x^2).\tag{4.27}$$

The energy shift is given by

$$\delta E = \frac{\sqrt{\lambda}}{2\pi} G'_{\text{mom}}(0).\tag{4.28}$$

4.3 Rank-One Sectors

We will now investigate the cases when only one of the physical resolvents is non-zero. The final result depends on the type of resolvent, \tilde{G} , \hat{G} or G^* .

Bosonic, Compact. We turn on only $G = \tilde{G}_{23}$ and consider one quasi-momentum $p = \tilde{p}_2 = -\tilde{p}_3$. This reduces to the case of strings on $\mathbb{R} \times S^3$ investigated in [17]

$$2\check{p}(x) = +2\mathcal{H}(x) + 2\tilde{F}(x) = 2\mathcal{G}(x) + \frac{2G'(0)/x}{1 - 1/x^2} + \frac{4\pi L}{\sqrt{\lambda}} \frac{1/x}{1 - 1/x^2} = -2\pi n_a \quad \text{for } x \in \mathcal{C}_a.\tag{4.29}$$

Note that the term $G(1/x)$ has precisely cancelled out and $p(x)$ is no longer symmetric under $x \mapsto 1/x$. This is related to the fact that spacetime is now a group manifold. In this case, the image under inversion is given by a different quasi-momentum, here \tilde{p}_1 . Also the length L now becomes a true global charge next to J . This is related to the left and right symmetry of group manifolds.

Bosonic, Non-Compact. We turn on only $G = \hat{G}_{14}$ and consider the single quasi-momentum $p = \hat{p}_1 = -\hat{p}_4$. This reduces to the case of strings on $AdS_3 \times S^1$ investigated in [28]

$$2\check{p}(x) = -2\check{H}(x) + 2\hat{F}(x) = -2\mathcal{G}(x) - \frac{2G(0)/x^2}{1 - 1/x^2} + \frac{4\pi L}{\sqrt{\lambda}} \frac{1/x}{1 - 1/x^2} = -2\pi n_a \quad \text{for } x \in \mathcal{C}_a. \quad (4.30)$$

Fermionic. We turn on only $G = G_{24}^*$ and consider the quasi-momenta $\tilde{p} = \tilde{p}_2$, $\hat{p} = \hat{p}_4$. The two quasi-momenta are given by $\tilde{p}(x) = H(x) + \tilde{F}(x) + F^*(x)$ and $\hat{p}(x) = H(x) - \hat{F}(x) + F^*(x)$

$$\begin{aligned} \tilde{p}(x) &= G(x) + \left(\frac{2\pi(B+L)}{\sqrt{\lambda}} + G'(0) \right) \frac{1/x}{1 - 1/x^2}, \\ \hat{p}(x) &= G(x) + \frac{2\pi(B-L)}{\sqrt{\lambda}} \frac{1/x}{1 - 1/x^2} + \frac{G(0)/x^2}{1 - 1/x^2}. \end{aligned} \quad (4.31)$$

The relevant combination for the integral equation is the difference of sheets $\tilde{p}(x) - \hat{p}(x) = \tilde{F}(x) + \hat{F}(x)$

$$\tilde{p}(x) - \hat{p}(x) = -\frac{G(0)/x^2}{1 - 1/x^2} + \frac{G'(0)/x}{1 - 1/x^2} + \frac{4\pi L}{\sqrt{\lambda}} \frac{1/x}{1 - 1/x^2} = -2\pi n_a \quad \text{for } x = x_a^*. \quad (4.32)$$

This agrees precisely with the expression derived from the near-plane-wave limit in [24].

4.4 Superstrings on $AdS_3 \times S^3$

The above three subsectors can be combined into one larger sector. Let us consider only the following four sheets $p_1 = \hat{p}_1$, $p_2 = \tilde{p}_2$, $p_3 = \tilde{p}_3$, $p_4 = \hat{p}_4$ and corresponding resolvents $\hat{G}_{41}, \tilde{G}_{23}, G_{21}^*, G_{31}^*, G_{24}^*, G_{34}^*$ so that again there is no apparent inversion symmetry. By inspection of the quasi-momenta there are only three independent combinations of resolvents appearing:

$$\begin{aligned} G_1 &= -G_{21}^* - G_{31}^* && -\hat{G}_{41}, \\ G_{\text{mom}} &= G_2 = +\tilde{G}_{23} - G_{31}^* + G_{24}^* && -\hat{G}_{41}, \\ G_3 &= +G_{34}^* && + G_{24}^* - \hat{G}_{41}. \end{aligned} \quad (4.33)$$

The quasi-momenta then read

$$\begin{aligned}
p_1(x) &= -G_1(x) + \frac{1/x}{1-1/x^2} \left(\frac{2\pi(B+L)}{\sqrt{\lambda}} + G'_2(0) - G'_1(0) \right) - \frac{G_1(0)/x^2}{1-1/x^2}, \\
p_2(x) &= G_2(x) - G_1(x) + \frac{1/x}{1-1/x^2} \left(\frac{2\pi(B+L)}{\sqrt{\lambda}} + G'_2(0) - G'_1(0) \right) - \frac{G_1(0)/x^2}{1-1/x^2}, \\
p_3(x) &= G_3(x) - G_2(x) + \frac{1/x}{1-1/x^2} \left(\frac{2\pi(B-L)}{\sqrt{\lambda}} + G'_3(0) - G'_2(0) \right) + \frac{G_3(0)/x^2}{1-1/x^2}, \\
p_4(x) &= G_3(x) + \frac{1/x}{1-1/x^2} \left(\frac{2\pi(B-L)}{\sqrt{\lambda}} + G'_3(0) - G'_2(0) \right) + \frac{G_3(0)/x^2}{1-1/x^2}.
\end{aligned} \tag{4.34}$$

The differences of adjacent sheets which appear in the integral equations are given by

$$\begin{aligned}
p_1(x) - p_2(x) &= -G_2(x), \\
p_2(x) - p_3(x) &= +2G_2(x) + \frac{2G'_2(0)/x}{1-1/x^2} + \frac{4\pi L}{\sqrt{\lambda}} \frac{1/x}{1-1/x^2} \\
&\quad - \tilde{G}_1(x) - \frac{G'_1(0)/x}{1-1/x^2} - \frac{G_1(0)/x^2}{1-1/x^2} \\
&\quad - G_3(x) - \frac{G'_3(0)/x}{1-1/x^2} - \frac{G_3(0)/x^2}{1-1/x^2}, \\
p_3(x) - p_4(x) &= -G_2(x).
\end{aligned} \tag{4.35}$$

Differences of non-adjacent sheets are obtained by summing up the equations.

For purely bosonic solutions on $AdS_3 \times S^3$ we set $G_1(x) = G_3(x)$ and

$$\begin{aligned}
\tilde{G}(x) = \tilde{G}_{23}(x) &= \int_{\tilde{C}} \frac{dy \tilde{\rho}(y)}{1-1/y^2} \frac{1}{y-x} = G_2(x) - G_1(x), \\
\hat{G}(x) = \hat{G}_{14}(x) &= \int_{\hat{C}} \frac{dy \hat{\rho}(y)}{1-1/y^2} \frac{1}{y-x} = G_1(x) = G_3(x).
\end{aligned} \tag{4.36}$$

The densities satisfy the following set of equations on the respective cuts:

$$\begin{aligned}
2\tilde{G}(x) + F(x) &= -2\pi\tilde{n}_a \quad \text{for } x \in \tilde{C}_a, \\
2\hat{G}(x) - F(x) &= -2\pi\hat{n}_a \quad \text{for } x \in \hat{C}_a,
\end{aligned} \tag{4.37}$$

where the potential $F(x)$ is given by

$$F(x) = \left(\frac{4\pi L}{\sqrt{\lambda}} + 2\tilde{G}'(0) \right) \frac{1/x}{1-1/x^2} - \frac{2\hat{G}(0)/x^2}{1-1/x^2}. \tag{4.38}$$

The momentum constraint (4.25) and energy shift (4.28) are contained in the combination

$$G_{\text{mom}}(x) = \tilde{G}(x) + \hat{G}(x). \tag{4.39}$$

4.5 Comparison to Gauge Theory

Let us briefly comment on the comparison to $\mathcal{N} = 4$ gauge theory. An in-depth comparison of the spectral curves can be found in [36]. The complete one-loop dilatation generator has been derived in [35, 15]. It is integrable and one can use a Bethe ansatz to find its energy (scaling dimension) eigenvalues [35]. In the thermodynamic limit [26, 10], which should be related to string theory [9], the Bethe equations have been written in [15]. Their form does not resemble the equations (4.21, 4.22, 4.24) very much, but rather the one in the previous Sec. 4.4. We shall refrain from transforming the equations here and refer the reader to App. D. The resulting equations (D.11)

$$\sum_{j'=1}^7 M_{j,j'} \#_{j'}(x) + F_j(x) = -2\pi n_{j,a} \quad \text{for } x \in \mathcal{C}_{j,a} \quad (4.40)$$

can be seen to agree with the equations in [15]. Also the expressions for the momentum constraint and energy shift as well as the local and global charges agree. Note that in the Frolov-Tseytlin limit, see Sec. 3.7, the potential F_j reduces to a term proportional to $V_j L/u$, c.f. [17, 29] for similar results. We have thus proven the agreement of the spectra of one-loop planar gauge theory and classical string theory.

5 Conclusions and Outlook

We solve the problem of describing all classical solutions of the superstring sigma-model in $AdS_5 \times S^5$ in terms of their spectral curves. Let us underline the importance of dealing with the whole supersymmetric string theory on the $AdS_5 \times S^5$ space, including the fermionic degrees of freedom, for its quantization. For the classical string we can drop the fermions and the two bosonic sectors, AdS_5 and S^5 , appear to be completely factorized (up to the constraints on the fixed poles and total momentum). Conversely, the quantum corrections at higher powers of $\hbar \sim 1/\sqrt{\lambda}$, make the two sectors interact nontrivially, an effect which is already seen in the super spin chain for the one-loop approximation to gauge theory. It is also clear that the direct quantization of sigma models in closed subsectors, like $\mathbb{R} \times S^5$, does not make much sense since those models are even not conformal. It seems that it is better to attack directly the full supersymmetric quantum theory on $AdS_5 \times S^5$. Our paper shows that, at least at the classical level, the full string theory has no principal difficulties comparing to the simpler subset sigma models.

The curves are solutions of a Riemann-Hilbert problem and as such can be encoded in the set of singular integral equations which we have derived. We hope that this classical result will be a useful starting point for a quantization. Some indications that the integrable structures persist in the quantum regime are found in [46–48, 31, 42, 49]. There are several benefits of the formulation in terms of algebraic curves which might facilitate quantization: For one, the formulation is completely gauge independent, at no point one is required to fix a gauge; especially we can preserve full kappa symmetry [34]. Moreover, the curve consists only of action variables. Due to the Heisenberg principle this is all we can ask for to know in the quantum theory. Finally, there are no unphysical degrees

of freedom associated to the curve. These would usually contain spurious infinities and their absence should make the curve completely finite. Our integral equations can be interpreted as the classical limit of the yet to be found discrete Bethe equations, which describe the exact quantum spectrum of the string, c.f. the ansatz by Arutyunov, Frolov and Staudacher [46] and a corresponding ‘string chain’ [47] for some initial steps in this direction. The existence of such equations is an assumption, but since many quantum integrable systems are solvable by a Bethe ansatz, in particular some sigma models [50], this assumption does not look inconceivable. We believe that in any event integrability will be an important ingredient in solving string theory in $AdS_5 \times S^5$, be it a Bethe ansatz or some other method, and hope that our findings will be helpful in attacking this challenging problem.

We should mention that many classical string solutions (those without self-symmetric cuts) admit a regular expansion in the ’t Hooft coupling.³² We have compared the energy spectrum of the classical string with the spectrum of anomalous dimensions in $\mathcal{N} = 4$ SYM which at one loop is given by a Bethe ansatz. Our comparison is based on Bethe equations but it can also be performed at the level of spectral curves. Although we should not expect agreement beyond third order in the effective coupling [12], we might use the present result as a source of inspiration for higher-loop gauge theory.

The method of solving the spectral problem in terms of algebraic curves which we employ for this sigma model is usually called the finite gap method (see the book [51] for a good introduction). This means that we look for a finite genus algebraic curve characterizing some solution. It was first proposed in [52] for the KdV system and later generalized to KP equations in [53]. In principle, for any given algebraic curve one can construct explicitly the corresponding solution of the equations of motion in terms of Riemann theta functions. The dependence on time enters linearly in the argument of the Riemann theta function and the frequencies are given by the periods of certain Abelian differentials. Note that in our investigation we have the angle variables, these enter as the initial phase for the arguments of the theta function. Moreover, to construct the solution for the $AdS_5 \times S^5$ coset model, one would first have to fix a gauge for the local symmetries. Finally, only up to genus one the theta functions can be expressed in terms of conventional algebraic and elliptic functions. Beyond that they are known only as integrals or series and therefore less efficient.

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³²Typically, this is an expansion in inverse powers of some large conserved charge. In our framework a useful conserved charge is the length L , see also [43]. The length is related to angular momentum J on S^5 in many cases and thus gives a natural generalization of the Frolov-Tseytlin proposal [9, 13].

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A Supermatrices

We shall write supermatrices as matrices where horizontal/vertical bars separate between rows/columns with even and odd grading. We shall only consider bosonic supermatrices. The grading matrix η consequently is given by

$$\eta = \left(\begin{array}{c|c} +I & 0 \\ \hline 0 & -I \end{array} \right). \quad (\text{A.1})$$

For example, it can be used to define the supertrace of a supermatrix as a regular trace of a supermatrix

$$\text{str } A = \text{tr } \eta A \quad (\text{A.2})$$

The supertrace is cyclic

$$\text{str } A B = \text{str } B A. \quad (\text{A.3})$$

The superdeterminant is defined as

$$\text{sdet} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \frac{\det(A - B D^{-1} C)}{\det D} = \frac{\det A}{\det(D - C A^{-1} B)}, \quad (\text{A.4})$$

it obeys

$$\text{sdet}(AB) = \text{sdet } A \text{sdet } B \quad (\text{A.5})$$

and is compatible with the identity

$$\text{sdet } \exp A = \exp \text{str } A. \quad (\text{A.6})$$

Supertranspose. The supertranspose is defined as

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)^{\text{ST}} = \left(\begin{array}{c|c} A^{\text{T}} & C^{\text{T}} \\ \hline -B^{\text{T}} & D^{\text{T}} \end{array} \right). \quad (\text{A.7})$$

Like the common transpose, it inverts the order within a product of matrices

$$(A B)^{\text{ST}} = B^{\text{ST}} A^{\text{ST}}. \quad (\text{A.8})$$

and does not change supertraces and superdeterminants

$$\text{str } A^{\text{ST}} = \text{str } A, \quad \text{sdet } A^{\text{ST}} = \text{sdet } A. \quad (\text{A.9})$$

Unlike the common transpose, it is not an involution, but a \mathbb{Z}_4 -operation due to the identity

$$(A^{\text{ST}})^{\text{ST}} = \eta A \eta \quad (\text{A.10})$$

Furthermore, a supersymmetric or a superantisymmetric matrix require a slightly modified definition

$$A = +\eta A^{\text{ST}}, \quad A = -\eta A^{\text{ST}}. \quad (\text{A.11})$$

(1|1) \times (1|1) Supermatrices. Let us collect some formulas for (1|1) \times (1|1) supermatrices

$$A = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array} \right) \quad (\text{A.12})$$

The inverse is given by

$$A^{-1} = \left(\begin{array}{c|c} \frac{1}{a} + \frac{bc}{a^2d} & -\frac{b}{ad} \\ \hline -\frac{c}{ad} & \frac{1}{d} - \frac{bc}{ad^2} \end{array} \right). \quad (\text{A.13})$$

The supertrace and superdeterminant read

$$\text{str } A = a - d, \quad \text{sdet } A = \frac{a}{d} - \frac{bc}{d^2}. \quad (\text{A.14})$$

It can be diagonalized by the matrices

$$T = \left(\begin{array}{c|c} 1 - \frac{bc}{2(a-d)^2} & +\frac{b}{a-d} \\ \hline -\frac{c}{a-d} & 1 + \frac{bc}{2(a-d)^2} \end{array} \right), \quad T^{-1} = \left(\begin{array}{c|c} 1 - \frac{bc}{2(a-d)^2} & -\frac{b}{a-d} \\ \hline +\frac{c}{a-d} & 1 + \frac{bc}{2(a-d)^2} \end{array} \right), \quad (\text{A.15})$$

such that

$$TAT^{-1} = \left(\begin{array}{c|c} \alpha_1 & 0 \\ \hline 0 & \alpha_2 \end{array} \right), \quad \alpha_1 = a + \frac{bc}{a-d}, \quad \alpha_2 = d + \frac{bc}{a-d}. \quad (\text{A.16})$$

The eigenvalues satisfy the sum and product rules

$$\alpha_1 - \alpha_2 = \text{str } A, \quad \frac{\alpha_1}{\alpha_2} = \text{sdet } A \quad (\text{A.17})$$

as well as the characteristic equation

$$\text{sdet}(A - \alpha_1) = 0, \quad \text{sdet}(A - \alpha_2) = \infty. \quad (\text{A.18})$$

Clearly α_1 and α_2 are associated to different gradings. For $\text{str } A = 0$, i.e. $a = d$, the eigenvalues degenerate and some problems arise as in the case of bosonic matrices.

B Cosets and Vectors

In this appendix we shall explain the relationship between the vector model used in [29] and the coset model used in this paper.

We will start with the coset model. The physically relevant cosets $S^5 = \text{SU}(4)/\text{Sp}(2)$ and $AdS^5 = \text{SU}(2, 2)/\text{Sp}(1, 1)$ require several i 's at various places. These can be avoided by considering the coset $\text{SL}(4, \mathbb{R})/\text{Sp}(4, \mathbb{R})$. Its algebraic structure is precisely the same but the formulas are slightly easier to handle. For the breaking of $\text{SL}(4, \mathbb{R})$ we can use a fixed 4×4 antisymmetric matrix E , say

$$E = \begin{pmatrix} 0 & +I \\ -I & 0 \end{pmatrix}, \quad (\text{B.1})$$

where each block corresponds to a 2×2 matrix and I is the identity. The currents of the coset model in the moving frame are

$$\begin{aligned} J &= -g^{-1}dg, \\ H &= \frac{1}{2}J - \frac{1}{2}EJ^T E^{-1}, \\ K &= \frac{1}{2}J + \frac{1}{2}EJ^T E^{-1}. \end{aligned} \quad (\text{B.2})$$

In the fixed frame, which is related to the moving frame by $j = gJg^{-1}$, etc., they are given by

$$\begin{aligned} j &= -dg g^{-1}, \\ h &= -\frac{1}{2}dg g^{-1} + \frac{1}{2}gEdg^T g^{-T} E^{-1} g^{-1}, \\ k &= -\frac{1}{2}dg g^{-1} - \frac{1}{2}gEdg^T g^{-T} E^{-1} g^{-1}. \end{aligned} \quad (\text{B.3})$$

We can now rewrite k as

$$k = -\frac{1}{2}dg g^{-1} - \frac{1}{2}gEdg^T (gEg^T)^{-1} = -\frac{1}{2}d(gEg^T) (gEg^T)^{-1}. \quad (\text{B.4})$$

and see that it can be rewritten as

$$k = -\frac{1}{2}dX X^{-1} \quad \text{with} \quad X = gEg^T. \quad (\text{B.5})$$

We would now like to interpret X as the fundamental field of the theory. For all $g \in \text{Sp}(4, \mathbb{R})$ we find $X = E$, thus X parametrizes the coset $\text{SL}(4, \mathbb{R})/\text{Sp}(4, \mathbb{R})$. Note that we can define a norm for X by

$$\varepsilon_{\alpha\beta\gamma\delta} X^{\alpha\beta} X^{\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta} E^{\alpha\beta} E^{\gamma\delta} \det g = -8. \quad (\text{B.6})$$

Starting from a generic matrix X , the conditions $X = -X^T$ and $\varepsilon_{\alpha\beta\gamma\delta} X^{\alpha\beta} X^{\gamma\delta} = -8$ leave 5 degrees of freedom for X and therefore such an X indeed parametrizes the coset $\text{SL}(4, \mathbb{R})/\text{Sp}(4, \mathbb{R})$ which has $15 - 10 = 5$ dimensions. We now parametrize X as

$$X = \vec{\sigma} \cdot \vec{X}, \quad \vec{X}^2 = -1. \quad (\text{B.7})$$

where \vec{X} is a vector of $\text{SO}(3, 3)$ and $\vec{\sigma}$ is a chiral component of the Clifford algebra. This reveals the connection between the sigma model (B.5) and the vector model (B.7), they are merely reparametrizations of the same model. More explicitly the matrix X is given through the components of the vector \vec{X} by

$$X = \begin{pmatrix} 0 & +X_1 + X_4 & +X_2 + X_5 & +X_3 + X_6 \\ -X_1 - X_4 & 0 & +X_3 - X_6 & -X_2 + X_5 \\ -X_2 - X_5 & -X_3 + X_6 & 0 & +X_1 - X_4 \\ -X_3 - X_6 & +X_2 - X_5 & -X_1 + X_4 & 0 \end{pmatrix} \quad (\text{B.8})$$

such that

$$\frac{1}{8}\varepsilon_{\alpha\beta\gamma\delta}X^{\alpha\beta}X^{\gamma\delta} = \vec{X}^2 = X_1^2 + X_2^2 + X_3^2 - X_4^2 - X_5^2 - X_6^2. \quad (\text{B.9})$$

The corresponding expressions for $\text{SO}(6)$ are

$$X = \begin{pmatrix} 0 & +X_1 + iX_2 & +X_3 + iX_4 & +X_5 + iX_6 \\ -X_1 - iX_2 & 0 & +X_5 - iX_6 & -X_3 + iX_4 \\ -X_3 - iX_4 & -X_5 + iX_6 & 0 & +X_1 - iX_2 \\ -X_5 - iX_6 & +X_3 - iX_4 & -X_1 + iX_2 & 0 \end{pmatrix} \quad (\text{B.10})$$

where again $\frac{1}{8}\varepsilon_{\alpha\beta\gamma\delta}X^{\alpha\beta}X^{\gamma\delta} = \vec{X}^2$, but with a positive signature of the norm. For $\text{SO}(2, 4)$ we find

$$X = \begin{pmatrix} 0 & +X_5 + iX_0 & +X_1 + iX_2 & X_3 + iX_4 \\ -X_5 - iX_0 & 0 & +X_3 - iX_4 & -X_1 + iX_2 \\ -X_1 - iX_2 & -X_3 + iX_4 & 0 & -X_5 + iX_0 \\ -X_3 - iX_4 & +X_1 - iX_2 & +X_5 - iX_0 & 0 \end{pmatrix} \quad (\text{B.11})$$

where \vec{X}^2 has signature $+- - - - +$. These expressions only differ from the expressions for $\text{SO}(3, 3)$ by relabelling the X_k and multiplying some of them by i .

C A Local Charge

Here we compute the first local charge as outlined in Sec. 2.6. Our starting point is (2.68) where we have already diagonalized the leading order \bar{A}_{-2} of the Lax connection using some matrix T_0 ,

$$\bar{A}_{-2} = \frac{1}{2}T_0(P_+ - \Lambda_\sigma)T_0^{-1} = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}. \quad (\text{C.1})$$

Let us see how this matrix can be used to block-diagonalize

$$\bar{A}_{-1} = \bar{X} + [T_1T_0^{-1}, \bar{A}_{-2}] \quad \text{with} \quad \bar{X} = T_0^{-1}Q_{1,\sigma}T_0 = \begin{pmatrix} u & v \\ x & y \end{pmatrix}. \quad (\text{C.2})$$

The key insight is that the double commutator

$$[\bar{A}_{-2}, [\bar{A}_{-2}, \bar{X}]] = (\alpha - \beta)^2 \begin{pmatrix} 0 & v \\ x & 0 \end{pmatrix} \quad (\text{C.3})$$

can be used to extract the off-diagonal elements. We can thus cancel them in \bar{A}_{-1} by setting

$$T_1 = \frac{1}{(\alpha - \beta)^2} [\bar{A}_{-2}, \bar{X}] T_0 \quad (\text{C.4})$$

and obtain

$$\bar{A}_{-1} = \bar{X} - \frac{1}{(\alpha - \beta)^2} [\bar{A}_{-2}, [\bar{A}_{-2}, \bar{X}]] = \begin{pmatrix} u & 0 \\ 0 & y \end{pmatrix}. \quad (\text{C.5})$$

We can continue and block-diagonalize \bar{A}_r order by order in this fashion. Note that A_r is block-diagonal if and only if

$$[\bar{A}_{-2}, \bar{A}_r] = 0. \quad (\text{C.6})$$

Together with the identity for any matrix \bar{Y}

$$[\bar{A}_{-2}, [\bar{A}_{-2}, [\bar{A}_{-2}, \bar{Y}]]] = (\alpha - \beta)^2 [\bar{A}_{-2}, \bar{Y}] \quad (\text{C.7})$$

one can construct the higher order transformation matrices quite conveniently. The local charges are defined via the trace of only one block a_r of \bar{A}_r . Again this can be achieved using the matrix \bar{A}_{-2} as follows

$$\frac{1}{\alpha - \beta} \text{str } \bar{A}_{-2} \bar{A}_r = \frac{1}{\alpha - \beta} (\alpha \text{str } a_r + \beta \text{str } b_r) = \text{str } a_r. \quad (\text{C.8})$$

Here it is important that $\text{str } \bar{A}_r = \text{str } a_r - \text{str } b_r = 0$.

Finally, we would like to express the local charges in terms of the physical currents $P, Q_{1,2}$. Note that all the expressions occurring in the conjugated $T_0^{-1} \bar{A}_r T_0$ are commutators of the currents, e.g.

$$T_0^{-1} \bar{A}_{-1} T_0 = Q_{1,\sigma} - \Delta_+^{-2} [P_+, [P_+, Q_{1,\sigma}]]. \quad (\text{C.9})$$

where $\Delta_+ = 2(\alpha - \beta)$ is the difference of eigenvalues of P_+ . The only exception is a term related to the diagonalization using T_0 , i.e. $T_0^{-1} \partial_\sigma T_0$. Within the trace they can be eliminated by making use of

$$[\bar{A}_{-2}, \partial_\sigma \bar{A}_{-2}] = 0 \quad (\text{C.10})$$

which is equivalent to the statement that $\partial_\sigma \bar{A}_{-2}$ is block-diagonal. This leads to

$$[P_+, [P_+, T_0^{-1} \partial_\sigma T_0]] = [P_+, \partial_\sigma P_+] \quad (\text{C.11})$$

and is sufficient to write every instance of $T_0^{-1} \partial_\sigma T_0$ within $\text{str } a_r$ in terms of $\partial_\sigma P_+$. Putting everything together we find

$$\begin{aligned} \text{str } a_2 = & \frac{1}{2} \Delta_+^{-1} \text{str } P_+ P_- + \Delta_+^{-5} \text{str} [P_+, D_\sigma P_+] [P_+, D_\sigma P_+] \\ & - 6 \Delta_+^{-5} \text{str} [[P_+, Q_{1,\sigma}], Q_{1,\sigma}] [P_+, D_\sigma P_+] \\ & + 2 \Delta_+^{-3} \text{str} [P_+, Q_{1,\sigma}] D_\sigma Q_{1,\sigma} \\ & - 2 \Delta_+^{-3} \text{str} [P_+, Q_{1,\sigma}] [P_+, Q_{2,\sigma}] \\ & - \Delta_+^{-5} \text{str} [[P_+, Q_{1,\sigma}], Q_{1,\sigma}] [[P_+, Q_{1,\sigma}], Q_{1,\sigma}] \\ & - 5 \Delta_+^{-7} \text{str} [[[P_+, Q_{1,\sigma}], P_+], Q_{1,\sigma}] [[[P_+, Q_{1,\sigma}], P_+], Q_{1,\sigma}]. \end{aligned} \quad (\text{C.12})$$

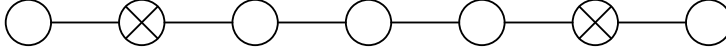


Figure 6: ‘Beauty’ Dynkin diagram of $\mathfrak{su}(2, 2|4)$ [22].

Here $D_\sigma X = \partial_\sigma X - [H_\sigma, X]$ is the world-sheet covariant derivative. The conserved charge corresponding to $\text{str } a_2$ is the integral

$$q_2^+ = -i \int_0^{2\pi} d\sigma \text{ str } a_2. \quad (\text{C.13})$$

Furthermore there exists a world-sheet parity conjugate charge q_2^- from the expansion around $z = \infty$ instead of $z = 0$. It is obtained from q_2^+ with the replacements $P_\pm \rightarrow -P_\mp$ and $Q_{1,2} \rightarrow Q_{2,1}$.

D Sleeping Beauty

This appendix contains lengthy expressions related to the complete superalgebra using the ‘Beauty’ form of $\mathfrak{psu}(2, 2|4)$ [22], c.f. Fig. 6. In this form, the grading of the sheets corresponding to the fundamental representation reads

$$\eta_k = (-1, -1, +1, +1, +1, +1, -1, -1). \quad (\text{D.1})$$

The sheets of the quasi-momentum are arranged as follows

$$p_{1,2,7,8} = \hat{p}_{1,2,3,4}, \quad p_{3,4,5,6} = \tilde{p}_{1,2,3,4}. \quad (\text{D.2})$$

D.1 Global Charges

The global fillings are defined as

$$K_j = \sum_{a=1}^A \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{C}_a} dx \left(1 - \frac{1}{x^2} \right) \sum_{k=1}^j \eta_k p_k(x). \quad (\text{D.3})$$

The global filling K_j essentially measures the total filling of all (k, l) -cuts with $k \leq j < l$. The global fillings are directly related to the Dynkin labels $[r_1; r_2; r_3, r_4, r_5; r_6; r_7]$ of a solution. The Dynkin labels are obtained through the residues at infinity

$$\bar{\eta}_j r_j = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\infty} dx (p_j(x) - p_{j+1}(x)), \quad (\text{D.4})$$

where $\bar{\eta}_j = [-1; +1; +1, +1, +1; +1; -1]$ are conventional factors for the definition of the Dynkin labels. These are given by

$$\begin{aligned}
r_1 &= K_2 - 2K_1, \\
r_2 &= K_3 - K_1 + \frac{1}{2}\delta E, \\
r_3 &= K_2 + K_4 - 2K_3, \\
r_4 &= L - 2K_4 + K_3 + K_5, \\
r_5 &= K_4 + K_6 - 2K_5, \\
r_6 &= K_5 - K_7 + \frac{1}{2}\delta E, \\
r_7 &= K_6 - 2K_7.
\end{aligned} \tag{D.5}$$

or for short

$$\bar{\eta}_j r_j = V_j L + \bar{V}_j \delta E - M_{j,j'} K_{j'} \tag{D.6}$$

The labels $\bar{V}_j = [0; \frac{1}{2}; 0, 0, 0; \frac{1}{2}; 0]$ indicate the change of (fermionic) Dynkin labels induced by the energy shift. Note that the Dynkin labels obey the central charge constraint

$$-r_1 + 2r_2 + r_3 = r_5 + 2r_6 - r_7. \tag{D.7}$$

The inverse relation is given by

$$\begin{aligned}
K_1 &= -\frac{1}{2}L + \frac{1}{2}B - \frac{1}{4}r^* - \frac{3}{4}r_1 + \frac{1}{2}r_2 + \frac{1}{2}r_3 + \frac{1}{2}r_4 + \frac{1}{2}r_5 + \frac{1}{2}r_6 - \frac{1}{4}r_7 - \frac{1}{2}\delta E, \\
K_2 &= -L + B - \frac{1}{2}r^* - \frac{1}{2}r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - \frac{1}{2}r_7 - \delta E, \\
K_3 &= -\frac{1}{2}L + \frac{1}{2}B - \frac{1}{4}r^* - \frac{1}{2}r_1 + r_2 + \frac{1}{4}r_3 + \frac{1}{2}r_4 + \frac{3}{4}r_5 + r_6 - \frac{1}{2}r_7 - \delta E, \\
K_4 &= -\frac{1}{2}r_1 + r_2 + \frac{1}{2}r_3 + \frac{1}{2}r_5 + r_6 - \frac{1}{2}r_7 - \delta E, \\
K_5 &= -\frac{1}{2}L - \frac{1}{2}B + \frac{1}{4}r^* - \frac{1}{2}r_1 + r_2 + \frac{3}{4}r_3 + \frac{1}{2}r_4 + \frac{1}{4}r_5 + r_6 - \frac{1}{2}r_7 - \delta E, \\
K_6 &= -L - B + \frac{1}{2}r^* - \frac{1}{2}r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - \frac{1}{2}r_7 - \delta E, \\
K_7 &= -\frac{1}{2}L - \frac{1}{2}B + \frac{1}{4}r^* - \frac{1}{4}r_1 + \frac{1}{2}r_2 + \frac{1}{2}r_3 + \frac{1}{2}r_4 + \frac{1}{2}r_5 + \frac{1}{2}r_6 - \frac{3}{4}r_7 - \frac{1}{2}\delta E.
\end{aligned} \tag{D.8}$$

The constant B represents the hypercharge of the vacuum.

D.2 Integral Representation

We present the reduction of a full set of resolvents into seven simple ones G_j for the $AdS_5 \times S^5$ superstring. The easiest way to reduce the expressions is to drop all but the resolvents between adjacent sheets. When the remaining resolvents are replaced by the suitably defined simple resolvents G_j

$$\begin{aligned}
G_1 &= -\hat{G}_{21} + \dots, \\
G_2 &= -G_{12}^* + \dots, \\
G_3 &= +\tilde{G}_{12} + \dots, \\
G_{\text{mom}} = G_4 &= +\tilde{G}_{23} + \dots, \\
G_5 &= +\tilde{G}_{34} + \dots, \\
G_6 &= +G_{43}^* + \dots, \\
G_7 &= -\hat{G}_{43} + \dots,
\end{aligned} \tag{D.9}$$

the original expressions are recovered. The quasi-momenta in terms of simple resolvents read

$$\begin{aligned}
p_1(x) &= -H_1(x) + G_2(1/x) - \frac{G_2(0)}{1-1/x^2} + \frac{(c_1+d_1)/x}{1-1/x^2}, \\
p_2(x) &= H_1(x) - H_2(x) + G_2(1/x) - \frac{G_2(0)}{1-1/x^2} + \frac{(c_1+d_1)/x}{1-1/x^2}, \\
p_3(x) &= H_3(x) - H_2(x) + G_2(1/x) - \frac{G_2(0)}{1-1/x^2} + \frac{(c_1+d_1)/x}{1-1/x^2} - G_4(1/x) + G_4(0), \\
p_4(x) &= H_4(x) - H_3(x) + G_2(1/x) - \frac{G_2(0)}{1-1/x^2} + \frac{(c_1+d_1)/x}{1-1/x^2} - G_4(1/x) + G_4(0), \\
p_5(x) &= H_5(x) - H_4(x) - G_6(1/x) + \frac{G_6(0)}{1-1/x^2} + \frac{(c_1-d_1)/x}{1-1/x^2} + G_4(1/x) - G_4(0), \\
p_6(x) &= H_6(x) - H_5(x) - G_6(1/x) + \frac{G_6(0)}{1-1/x^2} + \frac{(c_1-d_1)/x}{1-1/x^2} + G_4(1/x) - G_4(0), \\
p_7(x) &= H_6(x) - H_7(x) - G_6(1/x) + \frac{G_6(0)}{1-1/x^2} + \frac{(c_1-d_1)/x}{1-1/x^2}, \\
p_8(x) &= H_7(x) - G_6(1/x) + \frac{G_6(0)}{1-1/x^2} + \frac{(c_1-d_1)/x}{1-1/x^2} \tag{D.10}
\end{aligned}$$

with $c_1 = 2\pi B/\sqrt{\lambda} + \frac{1}{2}G'_6(0) - \frac{1}{2}G'_2(0)$ and $d_1 = 2\pi L/\sqrt{\lambda} + G'_4(0) - \frac{1}{2}G'_2(0) - \frac{1}{2}G'_6(0)$. The integral equations are given by

$$\p_j(x) - \p_{j+1}(x) = - \sum_{j'=1}^7 M_{j,j'} \p_{j'}(x) - F_j(x) = 2\pi n_{j,a} \quad \text{for } x \in \mathcal{C}_{j,a} \tag{D.11}$$

with $M_{j,j'}$ the Cartan matrix. Here, the non-zero potentials $F_j(x)$ read

$$\begin{aligned}
F_2(x) &= F_6(x) = G_4(1/x) - G_4(0), \\
F_4(x) &= -2G_4(1/x) + 2G_4(0) + \frac{2G'_4(0)/x}{1-1/x^2} \\
&\quad + G_2(1/x) - \frac{G_2(0)}{1-1/x^2} - \frac{G'_2(0)/x}{1-1/x^2} \\
&\quad + G_6(1/x) - \frac{G_6(0)}{1-1/x^2} - \frac{G'_6(0)/x}{1-1/x^2} \\
&\quad + \frac{4\pi L}{\sqrt{\lambda}} \frac{1/x}{1-1/x^2}. \tag{D.12}
\end{aligned}$$

References

- [1] J. M. Maldacena, “*The large N limit of superconformal field theories and supergravity*”, Adv. Theor. Math. Phys. 2, 231 (1998), [hep-th/9711200](#). • S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “*Gauge theory correlators from non-critical string theory*”, Phys. Lett. B428, 105 (1998), [hep-th/9802109](#). • E. Witten, “*Anti-de Sitter space and holography*”, Adv. Theor. Math. Phys. 2, 253 (1998), [hep-th/9802150](#).

- [2] R. R. Metsaev and A. A. Tseytlin, “*Type IIB superstring action in $AdS_5 \times S^5$ background*”, Nucl. Phys. B533, 109 (1998), [hep-th/9805028](#).
- [3] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “*A new maximally supersymmetric background of IIB superstring theory*”, JHEP 0201, 047 (2002), [hep-th/0110242](#). • M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “*Penrose limits and maximal supersymmetry*”, Class. Quant. Grav. 19, L87 (2002), [hep-th/0201081](#).
- [4] D. Berenstein, J. M. Maldacena and H. Nastase, “*Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills*”, JHEP 0204, 013 (2002), [hep-th/0202021](#).
- [5] R. R. Metsaev, “*Type IIB Green-Schwarz superstring in plane wave Ramond- Ramond background*”, Nucl. Phys. B625, 70 (2002), [hep-th/0112044](#). • R. R. Metsaev and A. A. Tseytlin, “*Exactly solvable model of superstring in plane wave Ramond- Ramond background*”, Phys. Rev. D65, 126004 (2002), [hep-th/0202109](#).
- [6] C. G. Callan, Jr., H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson and X. Wu, “*Quantizing string theory in $AdS_5 \times S^5$: Beyond the pp-wave*”, Nucl. Phys. B673, 3 (2003), [hep-th/0307032](#).
- [7] A. Parnachev and A. V. Ryzhov, “*Strings in the near plane wave background and AdS/CFT* ”, JHEP 0210, 066 (2002), [hep-th/0208010](#). • C. G. Callan, Jr., T. McLoughlin and I. Swanson, “*Holography beyond the Penrose limit*”, Nucl. Phys. B694, 115 (2004), [hep-th/0404007](#). • C. G. Callan, Jr., T. McLoughlin and I. Swanson, “*Higher impurity AdS/CFT correspondence in the near-BMN limit*”, Nucl. Phys. B700, 271 (2004), [hep-th/0405153](#). • T. McLoughlin and I. Swanson, “ *N -impurity superstring spectra near the pp-wave limit*”, Nucl. Phys. B702, 86 (2004), [hep-th/0407240](#).
- [8] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “*A semi-classical limit of the gauge/string correspondence*”, Nucl. Phys. B636, 99 (2002), [hep-th/0204051](#). • S. Frolov and A. A. Tseytlin, “*Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* ”, JHEP 0206, 007 (2002), [hep-th/0204226](#). • J. G. Russo, “*Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$* ”, JHEP 0206, 038 (2002), [hep-th/0205244](#). • J. A. Minahan, “*Circular semiclassical string solutions on $AdS_5 \times S^5$* ”, Nucl. Phys. B648, 203 (2003), [hep-th/0209047](#). • A. A. Tseytlin, “*Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond*”, Int. J. Mod. Phys. A18, 981 (2003), [hep-th/0209116](#).
- [9] S. Frolov and A. A. Tseytlin, “*Multi-spin string solutions in $AdS_5 \times S^5$* ”, Nucl. Phys. B668, 77 (2003), [hep-th/0304255](#).
- [10] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, “*Stringing Spins and Spinning Strings*”, JHEP 0309, 010 (2003), [hep-th/0306139](#).
- [11] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, “*Precision Spectroscopy of AdS/CFT* ”, JHEP 0310, 037 (2003), [hep-th/0308117](#).
- [12] D. Serban and M. Staudacher, “*Planar $\mathcal{N} = 4$ gauge theory and the Inozemtsev long range spin chain*”, JHEP 0406, 001 (2004), [hep-th/0401057](#).
- [13] A. A. Tseytlin, “*Spinning strings and AdS/CFT duality*”, [hep-th/0311139](#), in: “*From Fields to Stings: Circumnavigating Theoretical Physics*”, Ian Kogan Memorial Volume, ed.: M. Shifman, A. Vainshtein and J. Wheeler, World Scientific (2005), Singapore.

- [14] A. A. Tseytlin, “*Semiclassical strings in $AdS_5 \times S^5$ and scalar operators in $\mathcal{N} = 4$ SYM theory*”, *Comptes Rendus Physique* 5, 1049 (2004), [hep-th/0407218](#).
- [15] N. Beisert, “*The Dilatation Operator of $\mathcal{N} = 4$ Super Yang-Mills Theory and Integrability*”, *Phys. Rept.* 405, 1 (2005), [hep-th/0407277](#).
- [16] N. Beisert, “*Higher-loop integrability in $\mathcal{N} = 4$ gauge theory*”, *Comptes Rendus Physique* 5, 1039 (2004), [hep-th/0409147](#). • K. Zarembo, “*Semiclassical Bethe ansatz and AdS/CFT* ”, *Comptes Rendus Physique* 5, 1081 (2004), [hep-th/0411191](#).
- [17] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, “*Classical/quantum integrability in AdS/CFT* ”, *JHEP* 0405, 024 (2004), [hep-th/0402207](#).
- [18] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “*Spinning strings in $AdS_5 \times S^5$ and integrable systems*”, *Nucl. Phys. B* 671, 3 (2003), [hep-th/0307191](#). • G. Arutyunov, J. Russo and A. A. Tseytlin, “*Spinning strings in $AdS_5 \times S^5$: New integrable system relations*”, *Phys. Rev. D* 69, 086009 (2004), [hep-th/0311004](#).
- [19] V. E. Zakharov and A. V. Mikhailov, “*Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique*”, *Sov. Phys. JETP* 47, 1017 (1978), in russian.
- [20] K. Pohlmeyer, “*Integrable Hamiltonian systems and interactions through quadratic constraints*”, *Commun. Math. Phys.* 46, 207 (1976). • M. Lüscher and K. Pohlmeyer, “*Scattering of massless lumps and nonlocal charges in the two-dimensional classical nonlinear sigma model*”, *Nucl. Phys. B* 137, 46 (1978). • E. Brezin, C. Itzykson, J. Zinn-Justin and J. B. Zuber, “*Remarks about the existence of nonlocal charges in two-dimensional models*”, *Phys. Lett. B* 82, 442 (1979). • H. Eichenherr and M. Forger, “*Higher local conservation laws for nonlinear sigma models on symmetric spaces*”, *Commun. Math. Phys.* 82, 227 (1981).
- [21] J. A. Minahan and K. Zarembo, “*The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills*”, *JHEP* 0303, 013 (2003), [hep-th/0212208](#).
- [22] N. Beisert and M. Staudacher, “*The $\mathcal{N} = 4$ SYM Integrable Super Spin Chain*”, *Nucl. Phys. B* 670, 439 (2003), [hep-th/0307042](#).
- [23] N. Beisert, C. Kristjansen and M. Staudacher, “*The dilatation operator of $\mathcal{N} = 4$ conformal super Yang-Mills theory*”, *Nucl. Phys. B* 664, 131 (2003), [hep-th/0303060](#). • N. Beisert, “*The $su(2/3)$ dynamic spin chain*”, *Nucl. Phys. B* 682, 487 (2004), [hep-th/0310252](#).
- [24] M. Staudacher, “*The factorized S-matrix of CFT/AdS* ”, *JHEP* 0505, 054 (2005), [hep-th/0412188](#).
- [25] N. Beisert, V. Dippel and M. Staudacher, “*A Novel Long Range Spin Chain and Planar $\mathcal{N} = 4$ Super Yang-Mills*”, *JHEP* 0407, 075 (2004), [hep-th/0405001](#).
- [26] B. Sutherland, “*Low-Lying Eigenstates of the One-Dimensional Heisenberg Ferromagnet for any Magnetization and Momentum*”, *Phys. Rev. Lett.* 74, 816 (1995).
- [27] M. Kruczenski, “*Spin chains and string theory*”, *Phys. Rev. Lett.* 93, 161602 (2004), [hep-th/0311203](#). • M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, “*Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains*”, *Nucl. Phys. B* 692, 3 (2004), [hep-th/0403120](#).

- [28] V. A. Kazakov and K. Zarembo, “*Classical/quantum integrability in non-compact sector of AdS/CFT*”, JHEP 0410, 060 (2004), [hep-th/0410105](#).
- [29] N. Beisert, V. A. Kazakov and K. Sakai, “*Algebraic curve for the SO(6) sector of AdS/CFT*”, [hep-th/0410253](#), to appear in Comm. Math. Phys..
- [30] S. Schäfer-Nameki, “*The algebraic curve of 1-loop planar $\mathcal{N} = 4$ SYM*”, Nucl. Phys. B714, 3 (2005), [hep-th/0412254](#).
- [31] G. Arutyunov and S. Frolov, “*Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$* ”, JHEP 0502, 059 (2005), [hep-th/0411089](#).
- [32] A. Mikhailov, “*Supersymmetric null-surfaces*”, JHEP 0409, 068 (2004), [hep-th/0404173](#).
- [33] R. Hernández and E. López, “*Spin chain sigma models with fermions*”, JHEP 0411, 079 (2004), [hep-th/0410022](#).
- [34] I. Bena, J. Polchinski and R. Roiban, “*Hidden symmetries of the $AdS_5 \times S^5$ superstring*”, Phys. Rev. D69, 046002 (2004), [hep-th/0305116](#).
- [35] N. Beisert, “*The Complete One-Loop Dilatation Operator of $\mathcal{N} = 4$ Super Yang-Mills Theory*”, Nucl. Phys. B676, 3 (2004), [hep-th/0307015](#).
- [36] N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, “*Complete Spectrum of Long Operators in $\mathcal{N} = 4$ SYM at One Loop*”, JHEP 0507, 030 (2005), [hep-th/0503200](#).
- [37] M. Hatsuda and K. Yoshida, “*Classical integrability and super Yangian of superstring on $AdS_5 \times S^5$* ”, Int. J. Mod. Phys. A19, 4715 (2004), [hep-th/0407044](#).
- [38] A. Das, J. Maharana, A. Melikyan and M. Sato, “*The algebra of transition matrices for the $AdS_5 \times S^5$ superstring*”, JHEP 0412, 055 (2004), [hep-th/0411200](#).
- [39] R. Kallosh, J. Rahmfeld and A. Rajaraman, “*Near horizon superspace*”, JHEP 9809, 002 (1998), [hep-th/9805217](#). • N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, “*Superstring theory on $AdS_2 \times S^2$ as a coset supermanifold*”, Nucl. Phys. B567, 61 (2000), [hep-th/9907200](#).
- [40] R. Roiban and W. Siegel, “*Superstrings on $AdS_5 \times S^5$ supertwistor space*”, JHEP 0011, 024 (2000), [hep-th/0010104](#).
- [41] N. Berkovits, “*BRST cohomology and nonlocal conserved charges*”, JHEP 0502, 060 (2005), [hep-th/0409159](#).
- [42] N. Berkovits, “*Quantum consistency of the superstring in $AdS_5 \times S^5$ background*”, JHEP 0503, 041 (2005), [hep-th/0411170](#).
- [43] A. Mikhailov, “*Notes on fast moving strings*”, [hep-th/0409040](#). • A. Mikhailov, “*Plane wave limit of local conserved charges*”, [hep-th/0502097](#).
- [44] N. Berkovits, C. Vafa and E. Witten, “*Conformal field theory of AdS background with Ramond-Ramond flux*”, JHEP 9903, 018 (1999), [hep-th/9902098](#). • L. Dolan and E. Witten, “*Vertex operators for AdS_3 background with Ramond-Ramond flux*”, JHEP 9911, 003 (1999), [hep-th/9910205](#). • R. R. Metsaev and A. A. Tseytlin, “*Superparticle and superstring in $AdS_3 \times S^3$ Ramond-Ramond background in light-cone gauge*”, J. Math. Phys. 42, 2987 (2001), [hep-th/0011191](#).
- [45] G. Arutyunov and M. Staudacher, “*Matching Higher Conserved Charges for Strings and Spins*”, JHEP 0403, 004 (2004), [hep-th/0310182](#). • G. Arutyunov and M. Staudacher,

- “Two-loop commuting charges and the string/gauge duality”*, hep-th/0403077, in: *“Lie Theory and its Applications in Physics V”*, Proceedings of the Fifth International Workshop, Varna, Bulgaria, 16-22 June 2003, ed.: H.-D. Doebner and V. K. Dobrev, World Scientific (2004), Singapore.
- [46] G. Arutyunov, S. Frolov and M. Staudacher, *“Bethe ansatz for quantum strings”*, JHEP 0410, 016 (2004), hep-th/0406256.
 - [47] N. Beisert, *“Spin chain for quantum strings”*, Fortsch. Phys. 53, 852 (2005), hep-th/0409054.
 - [48] I. Swanson, *“Quantum string integrability and AdS/CFT”*, Nucl. Phys. B709, 443 (2005), hep-th/0410282.
 - [49] N. Beisert, A. A. Tseytlin and K. Zarembo, *“Matching quantum strings to quantum spins: one-loop vs. finite-size corrections”*, Nucl. Phys. B715, 190 (2005), hep-th/0502173. • R. Hernández, E. López, A. Periañez and G. Sierra, *“Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings”*, JHEP 0506, 011 (2005), hep-th/0502188.
 - [50] A. M. Polyakov and P. B. Wiegmann, *“Theory of nonabelian Goldstone bosons in two dimensions”*, Phys. Lett. B131, 121 (1983). • A. M. Polyakov and P. B. Wiegmann, *“Goldstone fields in two-dimensions with multivalued actions”*, Phys. Lett. B141, 223 (1984). • L. D. Faddeev and N. Y. Reshetikhin, *“Integrability of the principal chiral field model in $(1+1)$ -dimension”*, Ann. Phys. 167, 227 (1986). • E. Ogievetsky, P. Wiegmann and N. Reshetikhin, *“The principal chiral field in two-dimensions on classical Lie algebras: The Bethe ansatz solution and factorized theory of scattering”*, Nucl. Phys. B280, 45 (1987).
 - [51] S. Novikov, S. V. Manakov, L. P. Pitaevsky and V. E. Zakharov, *“Theory of Solitons. The Inverse Scattering Method”*, Consultants Bureau (1984), New York, USA, 276p, Contemporary Soviet Mathematics.
 - [52] A. R. Its and V. B. Matveev, *“Schrödinger operators with finite-gap spectrum and N -soliton solutions of the Korteweg-de Vries equation”*, Theor. Math. Phys. 23, 343 (1975). • B. A. Dubrovin, M. V. B. and S. P. Novikov, *“Non-linear equations of Korteweg-de Vries type, finite zone linear operators, and Abelian varieties”*, Russ. Math. Surveys 31, 59 (1976).
 - [53] I. M. Krichever, *“Elliptic solutions of KP equations and integrable systems of particles”*, Funk. Anal. App. 14, 282 (1980).